Cartograms for Non-Political Redistricting

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ABSTRACT: This paper examines math-based strategies for redistricting a state into a number N of congressional districts. A cartogram-inspired continuous map transformation of the state map is used to produce a new map having uniform population density. The new uniform density map is then further transformed by a continuous area-preserving (and, hence, uniform-density preserving) transformation into a grid of N regular hexagons. The two transformations are both invertible, allowing us to map the pre-images of the hexagons back onto the actual map of the state and to choose suitable district boundaries that weave close to the hexagon edges.

KEYWORDS: Area adjustment, automatic spatial partitioning, compactness, connectedness, optimization, gerrymandering

Introduction

Every ten years a census measures the changes and redistribution of population throughout the United States; and individual States are required to re-partition themselves into an appropriate number of nearly equally populated districts. The group drawing the new district lines can create political advantage for themselves by grouping the opposition into districts that intentionally reduce their proportional representation. To prevent such gerrymandering, legislatures passed laws and judges tried to interpret those laws. Unfortunately, neither the lawmakers nor the law-interpreters had the mathematical wherewithal to define or to measure impartiality in the drawing of district boundaries. Therefore we propose doing a little preprocessing of the population data and the State map to give the re-districters a problem that they can solve uniquely and easily, leaving no room for political shenanigans on their part. Here is our logic: Dividing the State of Ohio into 18 congressional districts would be easier if the state consisted of 18 equal-sized identically-shaped equally populated regular hexagons, as illustrated in Figure 1 below.

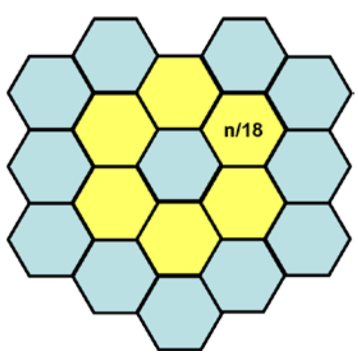

Figure 1. The new State of Ohio map with 18 identical equally-populated districts

Topology, sometimes called rubber-sheeting geometry, can be used to stretch and reshape any single connected region with no holes into any other simply connected shape; and, moreover, the stretching can be accomplished in such a way that population becomes uniformly re-distributed over the entire newly-shaped "State." Finally, the hexagon boundaries may be used to control the selection of tightly conforming linear map features chosen from preferred candidates for district boundary features.

We describe how to accomplish the topological transformation so that point displacements are minimized and how to keep legally-mandated significant geographic groupings together in the same districts as much as possible. We will develop topological coordinates based on population counts to redistribute the state population evenly to a new region and then further transform that new region into the region consisting of 18 hexagons. Finally we will use the inverse of the resulting cartogram transformation to reposition the hexagonal district boundaries on the original map.

Cartograms

Waldo Tobler has been studying cartograms for over 44 years (Tobler, 2004); and one of his key observations is that even though a cartogram is not technically a map projection, the most useful cartograms are projection-like. By projection-like, Tobler was referring to bicontinuous one-to-one correspondences of points on our datum surface and points on our map drawing. A one-to-one bicontinuous point correspondence allows one to perform point set operations like partitioning on one surface, say, the cartogram, and automatically induce equivalent point set operations on the other surface.

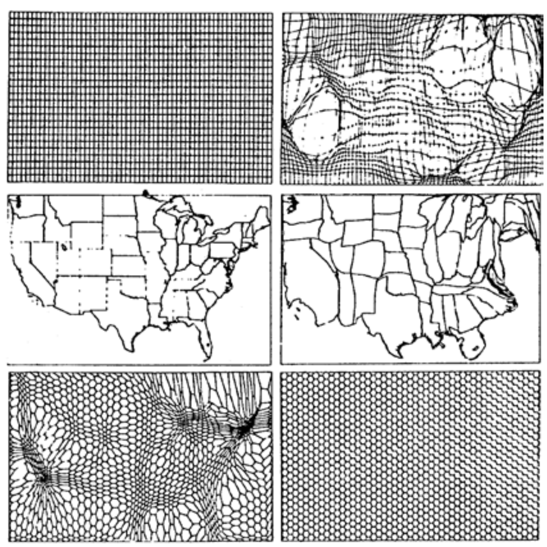

Fig. 2. Waldo Tobler's continuous deformation iterative procedure for equipopulous districts (1973).

Figure 2 shows an illustration by Tobler (1973) that illustrates hexagonal cookie-cutting of a cartogram with uniform population density, hence produces uniform sized areas all having the same population totals. Others have considered building spatial transformations that redistribute population uniformly. Rushton (1971) discussed stretching a rubber map with dots representing person locations to produce new dot positions that were on a regular grid or lattice. Gastner and Newman (2004) presented a diffusion-based model to simulate flows from higher density to lower density to arrive at uniform density. Kocmoud (1997) worked on a spring/energy balancing model on feature vertices with forces proportional to population density. His model could easily have been adapted to balance person separation within the regions.

Creating maps having equal population density everywhere

Saalfeld (2009) presented some applications of Schnyder's graph drawing constructions that turned population distributions into new spatial coordinates based on those distributions. These new coordinates necessarily created areas proportional to population counts; and although the coordinates arose from discrete counts, they could readily be interpreted as real-valued coordinates and interpolated linearly to non-integer values. In effect, the coordinates defined from integer person counts had natural extensions to non-integer-valued locations; and if the coordinates represented Euclidean locations, the space had uniform population density.

After weights are assigned to grid cells according to population counts in Figure 3, the grid structure is re-drawn with barycentric coordinates matching weighted-triangle-count coordinates, thus producing a map M_1 with uniform population density (see Fig. 4).

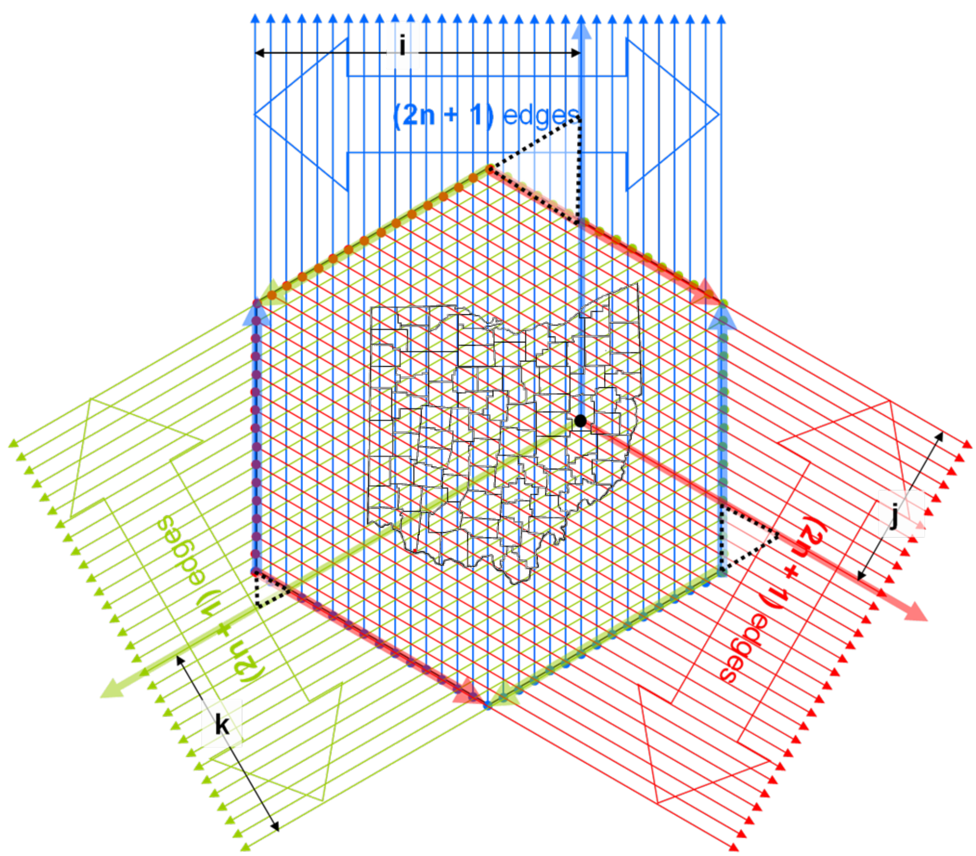

Fig 3. A triangular graph grid superimposed on a state map (cells to be weighted by population)

Figure 4 illustrates the result of using barycentric coordinates to create an "Ohio" as map M_1 with uniform population density everywhere.

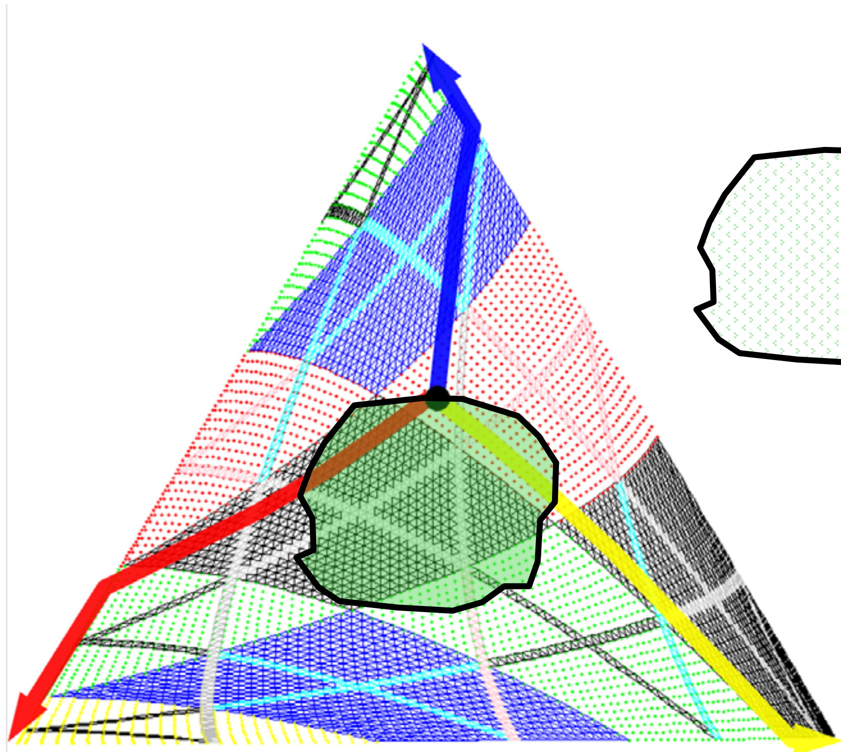

Fig. 4. The map M_1 with uniform population density produced by moving vertices to barycentric coordinates.

Arriving at a bijective bicontinuous transformation of the whole state to a region with uniform population density is just the first step in our redistricting strategy. Suppose M is an original map of the state. Let M_1 be the map cartogram of uniform population density that results from a bijection of points: $\beta: M \rightarrow M_1$.

After we have built a map M_1 of the state that redistributes the population uniformly so that population density is the same everywhere, we are going to further deform that uniform density map M_1 into another map M_2 so that (1) M_2 keeps its uniform population density, but now (2) M_2 has a new shape made up of N regular hexagons whose total area is the same as M_1.

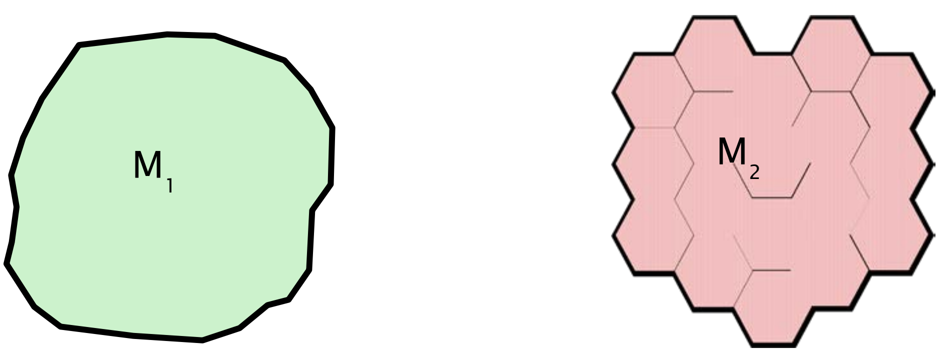

Fig. 5. Next we need an area preserving map into a region made up of the right number of hexagons

When region shapes are very similar as in Fig. 5, an area preserving function with very little displacement is easy to find.

An area-preserving map between two star-shaped regions of equal area

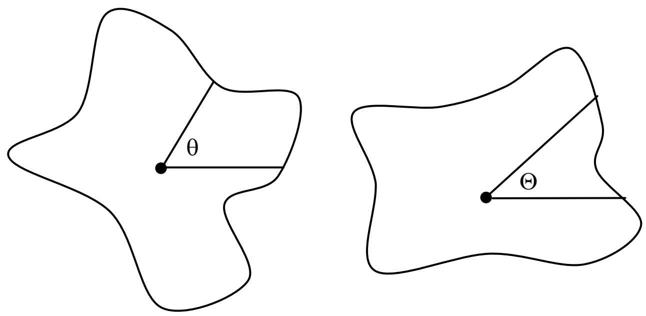

Fig. 6. Shapes to show that an area preserving function is easy to construct.

A region is star-shaped if from some vantage point called the star-center, all points in the boundary are visible and are not blocked from view by other points in the boundary (as can be seen in Fig. 6).

From the star-centers, for each θ in region 1, associate a Θ in region 2 that makes the area of region 2 subtended by the angle from 0° to Θ equal to the area of region 1 subtended by the angle from 0° to θ. Let $\pi(\theta)$ be the length of the segment from the star-center of region 1 to the boundary of region 1 in the direction θ; and let $\Pi(\Theta)$ be the length of the segment from the star-center of region 2 to the boundary of region 2 in the direction Θ.

This area rule defines the function $\theta \mapsto \Theta(\theta)$; and that area rule translates as two definite integrals being equal:

$$\tfrac{1}{2} \int \pi^2(\theta)\, d\theta = \tfrac{1}{2} \int \Pi^2(\Theta)\, d\Theta,$$

where the first integral goes from 0 to θ_0, and the second integral goes from 0 to $\Theta(\theta_0)$.

After computing $\Theta(\theta)$, we will scale points along the ray in region 2 in the direction Θ proportionally so that they match points along the ray in region 1 in the direction θ:

For a given θ fixed and any ρ between 0 and $\pi(\theta)$, define $P(\rho, \theta) = \rho\, \Pi(\Theta)/\pi(\theta)$.

Thus, we have completely defined the area-preserving: $(\rho, \theta) \mapsto (P(\rho, \theta), \Theta(\theta))$.

Weaving district boundaries on M_2

After drawing the hexagons, linear features transformed onto M_2 that cross a shared hexagon boundary may be used to exchange sliver regions of equal area (and hence equal population) between the hexagons in the map M_2. The sliver exchange should balance across each of the six hexagon segment boundaries since an exchange between those two regions must occur at their common boundary, as illustrated in Fig. 7. Coarser features such as county boundaries will likely force more area exchange between adjacent hexagons than finer features.

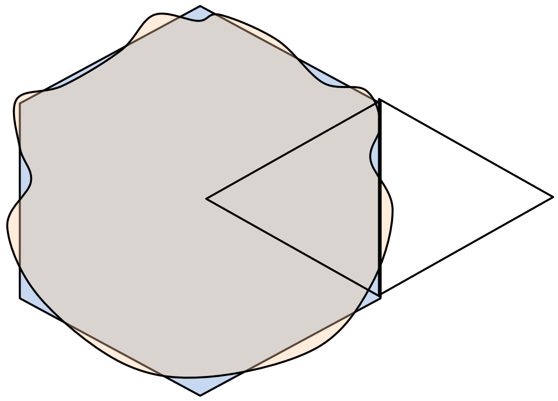

Fig. 7. On the map M_2, the shared edge between two hexagons may experience transfer of small equal slivers.

The three steps: spread population evenly over M_1; transform M_1 into a more desirable shape M_2; and swap equal sized slivers across individual hexagon edges of M_2, will identify districts that may convert easily to regions in the original map M.

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