# TOPOLOGICAL MODELS FOR 3D SPATIAL INFORMATION SYSTEMS

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## Abstract

The need for complex modelling and analysis of 3-dimensional data within a spatial information system (SIS) has been established in many fields. While much of the data that is currently being modelled seems to require "soft-edge" data structures such as grids or rasters, the need for certain types of complex topological modelling and analysis is clear. Current plane topology models such as the winged edge, widely used in computer aided design (CAD), are limited in the types of analysis that can be performed but useful because of their basis in the field of algebraic topology. This paper firstly reviews the neighborhood structure provided by current plane topological models. It then describes the derivation of a fundamental set of binary topological relationships between simple spatial primitives of like topological dimension in 3-space. It is intended that these relationships provide both a measure of modelling sufficiency and analytical ability in a spatial information system based on three dimensional neighborhoods.

## 1. Introduction

Modelling and analysis of 3-dimensional spatial phenomena has become a critical need in many applications, particularly the earth sciences. One of the traditional approaches to the <u>modelling</u> problem is to subset the sampled data from the 3D phenomena into individual spatial objects based upon theme or convenience; each spatial object can then be decomposed into a set of abstract geometric primitives - points, lines, faces and volumes; and a set of spatial relationships describing how the object may be reconstructed from these primitives. <u>Analysis</u> of the spatial phenomena requires not only the spatial relationships between the primitives required to reconstruct individual spatial objects, but also those relationships describing how the individual spatial objects interact. Such an approach is one method by which spatial objects may be modelled and analyzed according to theme or view in a larger model of the real phenomena.

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Topology is useful in both modelling and analysis because it provides simple and very useful spatial relationships, such as adjacency and connectivity. Topology can be thought of as the most primitive layer in a hierarchy of spatial relationships, where the next level of refinement is provided by the addition of familiar concepts based on a metric (ie. distance, direction etc.). Recent work by (Greasley 1988),(Kainz 1989) and (Kainz, 1990) in lattice theory seems to suggest that order relationships may exist at a similar level to topology.

Current topological models are either loosely or strongly based on a structure from algebraic topology known as the cell complex. The cell complex (in conjunction with graph theory) provides rules to govern the decomposition of a continuous 3D object into a finite number of points (0-cells), lines (1-cells), faces (2-cells) and volumes (3-cells). In governing the decomposition, the cell complex allows for the explicit description of three fundamental topological concepts: adjacency, connectivity and containment. Other relationships between individual objects such as whether two objects are disjoint or apart, may be provided by embedding individual cell complex(es) within a single cell or world cell and using the explicit relationships in combination to derive the particular relationship required. For example, it may be possible to analyze the explicit relationships to determine if two faces meet at a point (compare node connectivity of surrounding lines) or share a line (directly from adjacency). However, some relationships cannot be derived from these explicit relationships and may violate some of the rules of the cell complex. e.g. in 2-space ($R^2$) overlapping polygons (Egenhofer et. al. 1989); in 3-space ($R^3$), intersecting volumes or a face meeting another face at a point are all known to violate the rules of the cell complex governing the decomposition. In (Molenaar 1990) it is suggested that other relationships such as a line internal to a volume, also do not fit easily within the cell complex. From other work in 3D SIS (Youngmann 1988) and CAD (Weiler 1986), it appears that at least some of the modelling and analysis problems could be solved by combining the solid, surface and wire frame modelling approaches of CAD.

In this paper, the limitations of the cell complex are described by analyzing the direct and indirect topological relationships between cells that it provides. A layered set of fundamental binary topological relationships between simple lines, faces and volumes in $R^3$ based on point-set topology and extended from the work of (Egenhofer et. al. 1990) and (Pullar et. al. 1988) will be derived and presented. This paper and future research will attempt to integrate these intuitive yet powerful topological relationships and concepts with cell complex theory from algebraic topology since the power of the cell complex lies not in the nature and type of topological relationships that it allows, but in the ability to pose and solve topological problems as algebraic problems. It is expected that this approach will yield advantages both in modelling and analysis. For modelling purposes, the new topological relationships are intended to be used to ascertain the sufficiency of a cell complex based on 3D neighborhoods and provide insight into other useful structures such as lattices. Compactness and efficiency could be maintained by modelling only the coarsest topological relationships. For analysis purposes, a detailed set of basic topological relationships should provide either direct answers or at least the starting point of an answer, to complex spatial questions about the objects being modelled. In addition, enhancements to the fundamental modelling capability based on a complete set of topological relationships should allow boolean operations to be closed, i.e boolean operations may occur without the problem of not being able to model the result.

Sections 2 and 3 of this paper are concerned primarily with the cell complex. Section 2 introduces the necessary theory from algebraic topology and section 3 describes the current application of cell complex theory and the topological relationships which can be modelled. Section 4 introduces the necessary theory from point-set topology and presents the derivation of the new and richer set of topological relationships for $R^3$. Finally, section 5 concludes this paper with a summary of the results and the directions that will be taken in future research.

## 2. Topology

Topology is defined as the set of properties which are invariant under homeomorphisms (Alexandroff 1961) - one-to-one, continuous and onto transformations. Intuitively, it is easier to think of a homeomorphism as a kind of elastic transformation which twists, stretches and otherwise deforms without cutting. From the definition of topology as the study of those properties which remain invariant under homeomorphism, two objects are topologically equivalent if either can be transformed into the other using this type of elastic transformation. Clearly, metric properties such as distance, angle and direction are affected by homeomorphism and hence are not topological properties. It is the notion of homeomorphism which provides a fundamental or primitive set of spatial relationships (Chrisman 1987).

### About Neighborhoods

The neighborhood of a point is any open set (ie. a set that does not include its boundary) that contains the point. Neighborhoods can be defined in any abstract manner, but the most common are those that have a metric interpretation. For example in 2D, the neighborhood of a point can be considered as any 2D "flat" disk containing that point.

### About Manifolds

A manifold is an n-dimensional surface of which every point has a neighborhood topologically equivalent to an n-dimensional disk.This property is usually defined as local flatness. Manifolds are of interest because of their useful topological properties (in particular, the notion of orientation) which are inherited by the cells of a cell complex.

### About Simplexes, Cells and Complexes

An n-simplex is the n-dimensional simplest geometric figure eg. a 1-simplex is a line, a 2-simplex a triangle and a 3-simplex a tetrahedron - in essence, an n-simplex has n+1 vertices and may be viewed as the smallest closed convex set containing the given vertices (Alexandroff 1961). An n-simplex is the homeomorph of an n-cell. eg. any closed polygon which does not have an internal boundary (ie. genus 0) is homeomorphic to a triangle or 2-simplex. Because of this topological equivalence all results for simplexes generalize to cells.

An n-simplex is a composite of n-1,n-2,...,1 simplexes. eg. a 2-simplex or triangle, is bounded by three 1-simplexes, which meet at three 0-simplexes. In (Egenhofer et. al. 1989) this property is termed "Completeness of inclusion".

An n-simplicial complex or more generally an n-cell complex is the homeomorph of an n-dimensional polyhedron whose faces are all (n-1)-cells, no two of which intersect except at a cell of lower dimension. In (Egenhofer et. al. 1989) this intersection restriction is termed "Completeness of incidence". With this restriction, an n-cell complex may inherit the properties of an n-manifold, thus accessing the topological properties of manifolds, the most important of which is orientation. The notion of orientation is usually applied to the 1-simplex by defining one of the bounding 0-simplexes or points as a point of origin and the other as a point of termination. Relative orientations can then be assigned to all higher simplexes according to the traversal of bounding 1-simplexes.

## About Duality

Two dual operators which arise from these completeness axioms are termed boundary and coboundary, originally attributed to Poincaré (Corbett 1985). The boundary of an n-simplex is the incident set of n-1 simplexes. For example, a 3-simplex (tetrahedron) has 4 incident 2-simplexes, 6 incident 1-simplexes and 4 incident 0-simplexes. The coboundary of an n-simplex is the set of n+1-simplexes incident to the given n-simplex. For example, a 1-simplex may have two 2-simplexes cobounding it (one either side). The following table shows each cell and its dual, for $R^3$;

| Primal | Dual |
|---|---|
| 0-cell | 3-cell |
| 1-cell | 2-cell |
| 2-cell | 1-cell |
| 3-cell | 0-cell |

An important and powerful implication of duality is the fact that a primal may be represented and manipulated algebraically using its dual state. For example, 3-cells or volumes in a 3D SIS can be manipulated and represented by their dual state, the 0-cell or point.

## 3. Current Topological Models

Current topological models used either explicitly or implicitly in SIS and CAD fields are rather similar, despite the fact that CAD models are more generally used for 3D modelling and SIS are predominantly concerned with 2D models.

In SIS, 2D phenomena are assumed to be a connected set of points and lines (or a graph) which can be embedded in a 2-manifold - thus creating a set of connected and unconnected (or internal areas) (Corbett 1975),(Corbett 1979),(White 1983) and (White 1984). The application of the dual concepts of boundary and coboundary as described in the last section, provides connectivity and adjacency. Internal areas are described by simple application of homology theory. The data structures employed in such models are abstracted from graph theoretic concepts. Examples of systems built around these principles include DIME (Corbett 1975), ARC/INFO, TIGRIS (Herring, 1987), TIGER (Boudriault 1979).

In CAD and SIS surface modelling, even though 3D phenomena are being modelled, the current assumptions and resulting models are the same. The planar face of a 3D polyhedron is embedded in a 2-manifold and the embedded faces exist in 3D space resulting in a set of connected and unconnected (or internal) faces and volumes. The same application of the dual concepts of boundary and coboundary provides connectivity and adjacency e.g. (Corbett 1985) is a 3D extension of (Corbett 1975) and (Corbett 1979). Internal faces and volumes can be described by application of homology theory similar to that used for 2D SIS. The data structures employed in such models, such as the winged-edge model of (Baumgart 1975) and its later variants, e.g. (Braid et. al. 1978), (Woo 1985) and (Weiler 1985) are also based on graph theory and have been used extensively in CAD.

Both of these topological models can be described as vector, edge or boundary data structures and the particular topological relationships which are modelled can be classified using a system of relationships between 0,1 and 2D primitives specified in (Baer et. al. 1979) - see figure 1. Analysis of figure 1 shows that the main topological models in use, the winged-edge model in (Baumgart 1975) and the 2D map model in (Corbett 1975) and (Corbett 1979), both model the same set of relationships - EV and EF (from EV can derive VE, VV and EE, from EF can derive FE and FF, and from EF and EV together can derive VF and FV). Note that EV and EF give connectivity and adjacency corresponding with the boundary/coboundary principles of the cell complex - both models are basic applications of the cell complex. Most practical models do allow useful extensions that would normally be excluded by pure cell complex theory. For example, "dangling" lines - lines which are not connected at one or both ends to any other line.

The algebraic structure provided by cell complex theory has been reinvestigated in (Egenhofer et. al. 1989) and (Frank and Kuhn 1986) using concepts specified in (Giblin 1977) and (Moise 1977). This work has been applied to geological layers in (Carlson 1986) and to algorithms for editing triangular irregular networks in (Jackson 1989). The stated approach to the construction and maintenance of the cell complex is different to that taken previously because the construction and maintenance operations on the complex use topological concepts only - distance and other metric notions are not required.

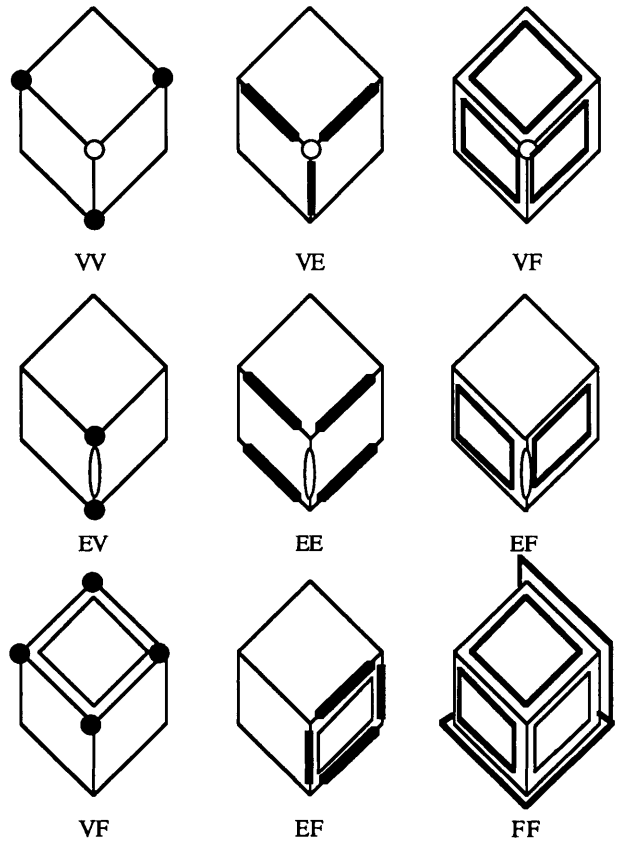



Figure 1 - 9 Relationships of (Baer, Henrion & Eastman 1979)

The intention is to avoid or at least minimize any inconsistency between the metric geometry and the topology that may be introduced by the limited precision arithmetic of computing devices (Franklin 1984). The other interesting aspect of (Egenhofer et. al.

1989) is that the construction and maintenance techniques are dimension independent.

In all models the necessary topological descriptions of faces with internal faces and volumes with internal volumes are described by the application of another branch of algebraic topology known as homology theory. e.g. (Corbett 1975), (Corbett 1979), (White 1983) and (Weiler 1985). Homology provides methods by which these internal faces and volumes may be detected by analysis of bounding cycles in cell complexes. In a wider sense homology groups give an indication of the connectivity present - internal faces and volumes may be regarded as homology group generators. In (Saalfeld 1989) other homology groups and additional homology theory are described and used in an attempt to determine the number of polygons resulting from the overlay of two maps.

## 4. Topological Relationships

What topological relationships may exist between abstract geometric primitives in euclidean 3-space? To answer a detailed question about the nature and type of all topological relationships is an attempt to classify the types and situations of manifolds. This is possible for $R^1$ (1-space) and $R^2$ (2-space) however, $R^3$ (3-space) has a number of quite difficult and unexpected situations which make general classification very difficult. See (Zeeman 1961) and (Alexandroff 1961). Fortunately, it is not necessary to attempt this. A number of assumptions about the nature of the relationships and the geometry of the n-cells involved can be made without limiting the power and application of the derived relationships. Specifically, only binary topological relationships between closed, connected (genus 0 - no internal holes) n-simplexes will be considered. The use of simplexes rather than cells is intuitive; simplicial complex theory is the starting point for the more generalized and advanced cell complex theory. Cells can be decomposed into simplexes in what is termed a simplicial decomposition, thus the results derived using simplicial complex theory can be generalized to cell complex theory via the decomposition.

In section 3, it was shown that cell complex theory as it is currently implemented in plane topology models allows a number of useful topological relationships such as adjacency and connectivity. In effect, cell complex theory allows n-dimensional adjacency (= connectivity in $R^1$), containment and the complement relationship of disjoint existing where no adjacency can be found. In essence, the main function of the cell complex is to allow specification of topological problems using algebraic methods, the definition of the algebraic operations being confined by the intersection rules (the set of allowable topological problems).

Point-set topology (classical topology) provides a much more intuitive view of topological relationships. In this paper, point-set binary topological relationships between 1-simplexes in $R^3$, 2-simplexes in $R^3$ and 3-simplexes in $R^3$ are based on consideration of the fundamental <u>boundary</u>, <u>interior</u> and <u>exterior</u> point-sets of any n-simplex in $R^n$. Additional point-sets are formed generically by embedding the n-simplex and its fundamental point-sets for $R^n$, within $R^{n+1}$. Consideration of the possible intersections of these point-sets with the boundary point-set of a second n-simplex then gives the fundamental topological relationships. The relationships are point-set topological relationships because they are derived from the intersection of these fundamental point-sets only.

The resulting binary topological relationships are very detailed. A number of methods can be chosen to aggregate or subdivide them into a hierarchy of detail. The method of aggregation chosen in this paper is consistent with the topological notion of homeomorphism. Each of the resulting binary topological relationships is considered to be the union of the two n-simplex point sets involved. Some topological relationships are then homeomorphic and can be replaced by a single homeomorph. The resulting tree structure then provides two levels of detail, the most descriptive relationships being found at the "leaves" of the tree. Further subdivision and grouping could also occur by considering the dimension of the spatial intersection between the two n-simplexes in each relationship as proposed in (Egenhofer et. al. 1990).

In all of the following discussion, a 1-simplex is called an interval, a 2-simplex is called a face and a 3-simplex is called a volume.

## Theoretical Background

All results used and derived in this section are for metric topological spaces since metric topological spaces are most commonly used for modelling purposes. Metric topological spaces are a subset of general topological spaces.

An n-simplex in $R^n$ divides $R^n$ into three useful and intuitive point-sets, well known in point-set topology; eg. (Kasriel 1971)

Interior $^\circ$ set of an n-simplex **C**: a point x is an interior point of **C** provided there exists an open subset **U** such that $x$ is an element of **U** and **U** is strictly contained within **C**. The union of all such points is the interior set.

Boundary set $\partial$ of an n-simplex **C**: $\mathbf{C} - {}^\circ$

Exterior set of an n-simplex **C**: Complement of **C**.

A simple and complete method can be found for finding all topological relationships between two closed, connected n-simplexes. In (Pullar et. al. 1988), (Driessen 1989) and (Egenhofer et. al. 1990) only the intersection of the boundary and the interior point-sets of the two n-simplexes is used to derive topological relationships. In this paper, a more powerful and fundamental method is used which is based on the set intersection of the boundary, interior and exterior point-sets of an n-simplex $n_1$ and the boundary, interior and exterior sets of another n-simplex $n_2$ in $R^n$. In practice, the derivation of relationships can be simplified by considering the possible set intersection of the boundary point-set of $n_1$ and the interior, exterior and boundary point-sets of $n_2$, since the boundary set of $n_1$ naturally defines the interior and exterior point-sets of $n_1$ and governs their possible relationships with the sets of $n_2$. Further detail can then be added to each relationship if required by considering the set intersection of the interior and exterior sets of $n_1$ with those of $n_2$.

Up till now the definitions of the fundamental point-sets of an n-simplex have been given in terms of an n-simplex in $R^n$, however in order to analyze intersections between n-simplexes in $R^{n+1}$ it is necessary to consider what happens to the simplex and its point-sets in $R^n$ when they are embedded in $R^{n+1}$. This is of particular importance to this research, since the aim is to derive topological relationships between 1-simplexes, 2-simplexes and 3-simplexes in $R^3$.

The closed boundary and open interior and exterior point-sets of an n-simplex in $R^n$ are all closed point-sets when considered relative to $R^{n+1}$ since the union of these point-sets is an n-manifold equivalent to $R^n$, and $R^n$ itself is a closed point-set in $R^{n+1}$. Since the intersection process is reliant upon the existence of these three point-sets then we have a problem, the solution to which can be found by considering the dimension of the n-manifold created from the union of these point-sets and the dimension of the space in which they to be embedded. In $R^n$, we are considering the intersection of the boundary, interior and exterior point-sets of two n-simplexes in the same n-manifold which is equivalent to $R^n$. In $R^{n+1}$, we consider not only the situation in $R^n$ where both n-simplexes are in the same n-manifold, but also the complement situation which occurs when both n-simplexes are in different n-manifolds. Clearly any intersection between the boundary, interior and exterior point-sets of the two n-simplexes will always occur where the two n-manifolds meet, hence if the open/closed point-set properties of the interior, exterior and boundary point-sets of an n-simplex are considered strictly relative to the n-manifold formed by their union, then their open/closed point-set properties are preserved and can be used without loss of generality regardless of the dimension of the space in which the n-manifold(s) created from their union are embedded.

It is now necessary to find a simple and comprehensive way of analyzing the intersection possibilities between two n-simplexes in $R^{n+1}$ excluding the subset formed specifically for $R^n$ when both n-simplexes are in the same n-manifold. This can be done by choosing a specific embedding of such an n-manifold or equivalently $R^n$, in $R^{n+1}$. If $R^n$ and $R^{n+1}$ are metric spaces with standard orthogonal basis vectors (or coordinate system axes) then if we choose the embedding such that the n orthogonal basis vectors of $R^n$ are coincident with n of the n+1 orthogonal basis vectors of $R^{n+1}$, then $R^n$ disconnects $R^{n+1}$ into two open point-sets corresponding to the opposing directions of the n+1th orthogonal basis vector of $R^{n+1}$. Using this fact the derivation method for possible intersections between n-simplexes in $R^{n+1}$ can be extended simply by considering those intersection combinations involving either or both of the two new point-sets resulting from the embedding.

In the summary of the theory and the rest of this paper, the generic term set is used in place of point-set. The theory can now be summarised in five steps as follows;

1. Formulate the boundary, interior and exterior sets of an n-simplex $n_1$ in $R^n$.
2. Derive basic relationships based on all possible set intersections of the boundary set of a second n-simplex $n_2$ and the interior, boundary and exterior sets of the n-simplex $n_1$ from step 1.
3. Consider the union of the interior, exterior and boundary sets of any n-simplex in $R^n$ as an n-manifold equivalent to $R^n$ with the definition of the open/closed properties of these sets strictly relative to $R^n$.
4. Disconnect $R^{n+1}$ into two new open sets by choosing an embedding of $R^n$ (created in step 3) in $R^{n+1}$ such that the n orthogonal basis vectors of $R^n$ are coincident with n of the n+1 orthogonal basis vectors of $R^{n+1}$.
5. Derive additional relationships based on the possible set intersections of the boundary set of an n-simplex $n_2$ with the boundary, interior and exterior sets of the a second n-simplex $n_1$ with the boundary set of $n_2$ intersecting either or both of the two new sets predicted in step 4.

Intervals (1-simplexes)

The boundary, interior and exterior sets of an interval $i_1$ in $R^1$ are shown in figure 2.

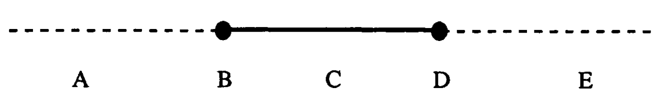



Figure 2 - The exterior, boundary and interior sets of an interval in R1

Note that there are two distinct closed boundary sets (B and D), two distinct open exterior sets (A and E) and a single open interior set (C). The union of the sets A,B,C,D and E is a 1-manifold equivalent to $R^1$. All possible binary topological relationships between two intervals in $R^1$ can then be derived by choosing any two points $x$ and $y$ forming the boundary set of a second interval $i_2$, either from the same set or each from a different set, and making these the boundaries of an interval joining them. The created interval $i_2$ will then either intersect interval $i_1$ in some way or be disjoint from it. e.g. If both points $x$ and $y$ are chosen from set A (the left exterior set) then the created interval $i_2$ will not intersect $i_1$. The unique combinations and their spatial interpretations are shown in figure 3.

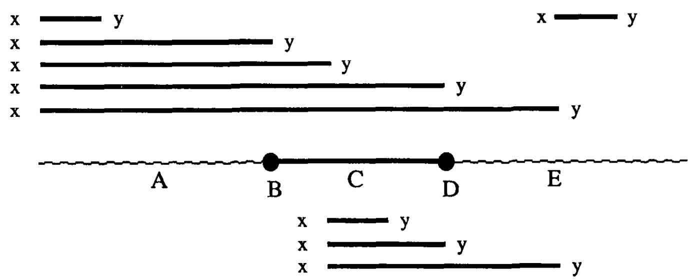



Figure 3 - Possible choice combinations of the two boundary points x,y from the boundary, interior and exterior sets of an interval in R1

From figure 3, it is possible to distinguish those choices which give distinct relationships and name these distinct relationships as follows; ($\varepsilon$ = element of a set)

| | | |
|---|---|---|
| $x$ $\varepsilon$ boundary set B or D, $y$ $\varepsilon$ boundary set D or B | -> | $i_1$ equals $i_2$ |
| $x,y$ $\varepsilon$ exterior set A or E | -> | $i_1$ and $i_2$ are disjoint |
| $x,y$ $\varepsilon$ interior set C | -> | $i_1$ contains $i_2$ |
| $x$ $\varepsilon$ exterior set A or E, $y$ $\varepsilon$ interior set C | -> | $i_1$ and $i_2$ overlap |
| $x$ $\varepsilon$ exterior set A or E, $y$ $\varepsilon$ boundary set B or D | -> | $i_1$ meets $i_2$ |
| $x$ $\varepsilon$ boundary set B or D, $y$ $\varepsilon$ interior set C | -> | $i_1$ and $i_2$ share common bounds |

Note that these six relationships are the same as those derived in (Pullar et. al. 1988). The names given to the six distinct relationships are also taken from (Pullar et. al. 1988).

If we define the open/closed properties of these sets strictly relative to $R^1$ then these sets and the set relationships in $R^1$ are preserved when the five sets A,B,C,D and E whose union comprises $R^1$ are embedded in $R^2$. As for the new sets created by the embedding; if the embedding of $R^1$ in $R^2$ is chosen such that the basis vector of $R^1$ corresponds to one of the two orthogonal basis vectors of $R^2$, then $R^2$ will be divided into two open sets F and G, separated by a third set corresponding to $R^1$. The situation is shown in figure 4. $R^1$ is represented by the line L.

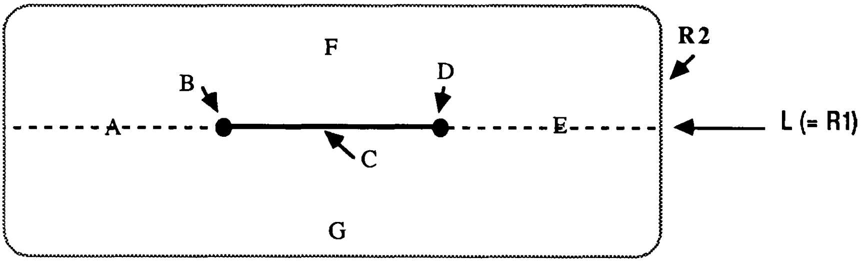



Figure 4 - New point sets F & G obtained by embedding R1 in R2

All possible binary topological relationships between intervals in $R^2$ can be derived in the same way as for $R^1$, by choosing two points either from the same set or from a different set and making these the boundary of an interval. Since those relationships derived in $R^1$ apply without modification in $R^2$, only the new combinations where x,y are elements of either or both sets F and G will be considered.

The set relationships can be divided into groups by examination of figure 4. The first group occurs when both boundary points are in the sets A,B,C,D or E which comprise $R^1$ (the line L) and has already been considered above. The second group occurs when either one or both of the boundary points of $i_2$ are contained within either F or G. The spatial situation corresponds to the interval $i_2$ being either left or right of the line L. The possible combinations and their spatial interpretations are shown in figure 5a.

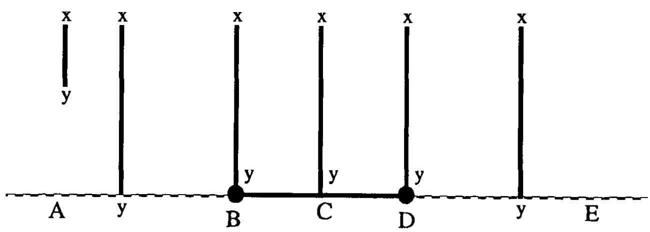



Figure 5a - Intersection between the boundary point y of interval i2 and the boundary, interior and exterior sets of interval i1 when boundary point x of i2 is always chosen from the point set F (or equivalently, G).

The following set relationships may be distinguished based upon which sets the boundary points x and y intersect;

$x \in$ set F, $y \in$ set F OR
$x \in$ set G, $y \in$ set F OR
$x \in$ set F or set G, $y \in$ exterior set A or E -> $i_1$ and $i_2$ disjoint

$x \in$ set F or set G, $y \in$ boundary set B or D -> $i_1$ meets $i_2$
$x \in$ set F or set G, $y \in$ interior set C -> $i_1$ intersects $i_2$

The third group occurs when one of boundary points of $i_2$ is an element of F and the other is an element of G, indicating that the interior set of interval $i_2$ intersects the line L (the boundary, exterior and interior sets of $i_1$) at a point. The possible combinations and their spatial interpretations are shown in figure 5b.

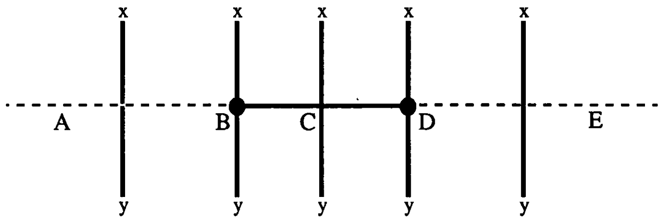



Figure 5b - Intersection between the interior set of interval i2 and the boundary, interior and exterior sets of interval i1 when x and y are chosen from the point sets F and G respectively.

The three resulting relationships are distinguished according to which set of $i_1$ that the interior of $i_2$ intersects (in fact, the three possible relationships between a single point and the interior, boundary and exterior sets of an interval);

$x \in$ set F, $y \in$ set G, intersect exterior set A or E -> $i_1$ and $i_2$ disjoint
$x \in$ set F, $y \in$ set G, intersect boundary set B or D -> $i_1$ intersects $i_2$
$x \in$ set F, $y \in$ set G, intersect interior set C -> $i_1$ and $i_2$ cross

By consideration of both these groups, the only new relationships which result are intersect and cross, making a total of 8 relationships between intervals in $R^2$. For $R^3$ also, no new relationships result because embedding the scheme for $R^2$ shown in figure 4, in $R^3$ produces two new sets as a result. The same process of reduction for $R^2$ reveals no new relationships - hence there are eight relationships between intervals in $R^3$.

To reduce these 8 relationships in detail, the union of the boundary and interior points-sets of $i_1$ and $i_2$ is considered. Relationships can then be eliminated which are homeomorphic. For intervals, this results in relationships; meet, overlap, contains, equal, common-bounds all being homeomorphic to a single interval. Thus, the complete two layer hierarchy of binary topological relationships between intervals (1-simplexes) in $R^3$ is shown in figure 6.

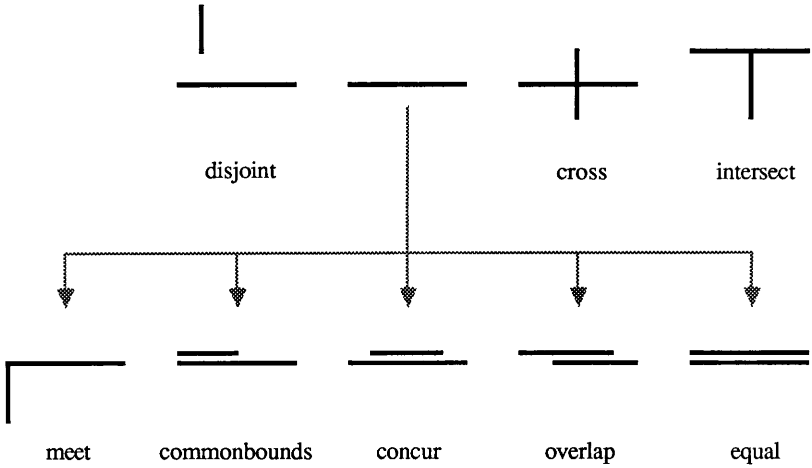



Figure 6 - The eight unique binary topological relationships between 1-cells in R3

Faces (2-simplexes)

The boundary, interior and exterior sets of a face (or 2-simplex) $a_1$ in $R^2$ are shown in figure 7.

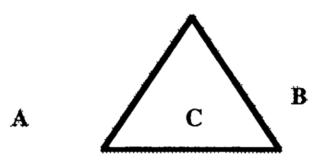



Figure 7 - Exterior (A), Boundary (B) and Interior (C) point-sets of a face in R2

Note that there is a single closed boundary set (B), a single open exterior set (A) and a single open interior set (C). The union of sets A,B and C is a 2-manifold equivalent to

$R^2$. All possible binary topological relationships between faces in $R^2$ can then be derived from the possible set relationships between the boundary, interior and exterior sets A, B and C of $a_1$ and the boundary set X of $a_2$. e.g. if the boundary set X of $a_2$ is contained within the interior set C, then the face $a_2$ will be contained within $a_1$. The combinations matrix showing the possible relationships between the boundary of the face $a_2$ and the exterior, boundary and interior sets A,B, and C of $a_1$ is shown in table 1.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Exterior A | X | | | X | | X | X |
| Boundary B | | X | | X | X | X | |
| Interior C | | | X | X | X | | X |

Table 1: Set intersection relationships between the boundary set of a2 and the interior, exterior and boundary sets of a1 in R2

Note that the seventh relationship in the last column of table 1 is not possible in $R^2$ because of the restriction to closed, connected faces.

The six distinct relationships and their names are the same as those in (Egenhofer et. al. 1990). The spatial interpretations are shown in figure 8.

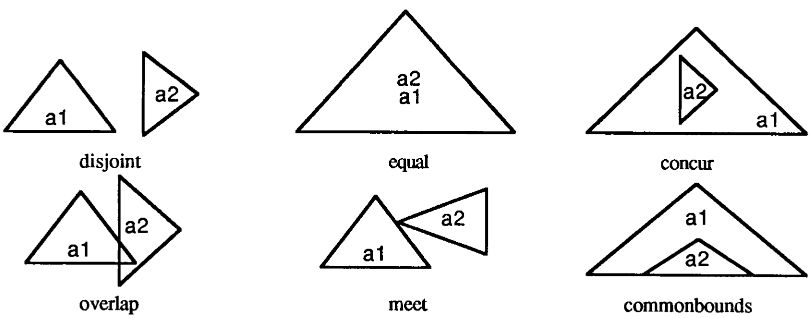



Figure 8 - Six possible relationships between faces based on the intersection of the boundary set of face a2 and the exterior, boundary and interior sets of face a1 in R2

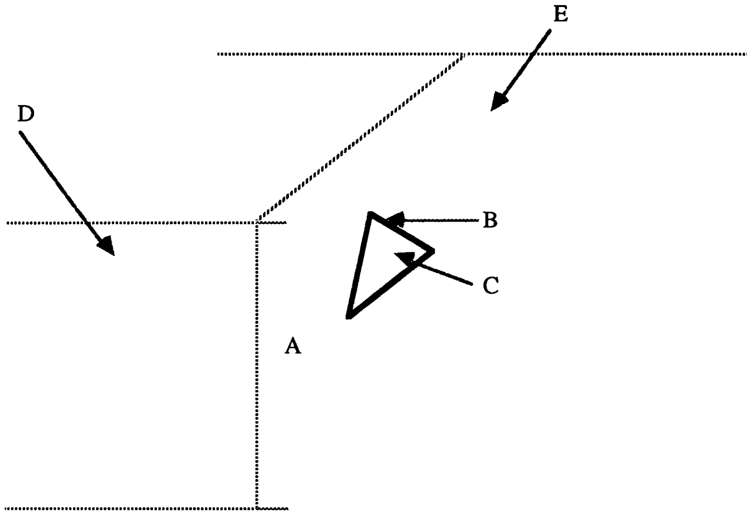



Figure 9 - New point sets D & E obtained by embedding the union of the boundary (B), exterior (A) and interior (C) of a face al in R3 (A U B U C = R2).

If we define the open/closed properties of these sets strictly relative to $R^2$ then these properties and the set relationships in $R^2$ are preserved when the 2-manifold (equivalent to $R^2$) formed by their union is embedded in $R^3$. If the embedding is chosen such that any two orthogonal basis vectors of $R^2$ are coincident to two of any three orthogonal basis vectors of $R^3$ then $R^2$ disconnects $R^3$ into two open sets with the third open set corresponding to $R^2$ itself. The situation is shown in figure 9.

All possible binary topological relationships between faces in $R^3$ can be derived in the same way as for $R^2$, by considering the possible set relationships between boundary set of a face $a_2$ and the boundary, interior and exterior sets of the face $a_1$ plus the two new sets D and E which result from embedding $R^2$ in $R^3$. Since all set relationships derived for $R^2$ are preserved in $R^3$, only the combinations involving the new sets D and E will be considered.

By examination of figure 9, the set relationships can be divided into two groups. The first group represents the situation where the boundary set X of $a_2$ is contained within the plane **P** formed from the union of the interior, exterior and boundary sets of $a_1$. This situation corresponds to faces in $R^2$ and was considered above. The second group corresponds to the situation where the boundary set X of $a_2$ intersects either D or E but not both. This corresponds to the spatial situation where $a_2$ is completely on one side of the plane **P** formed by the boundary, interior and exterior sets A,B and C of $a_1$. In this situation, the boundary set X of $a_2$ may intersect the plane **P** and hence the boundary, interior and exterior sets A,B and C or not at all. All combinations are shown in table 2.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Exterior A | | | | X | | X | X | X |
| Boundary B | | | X | | X | X | | X |
| Interior C | | X | | | X | | X | X |
| Above D<br>Below E | X | X | X | X | X | X | X | X |

Table 2: Set intersections between the boundary set X of a2
and the interior (A), exterior (B) and boundary (C) sets of a1 in R3 when a1
intersects only one of the sets D or E.

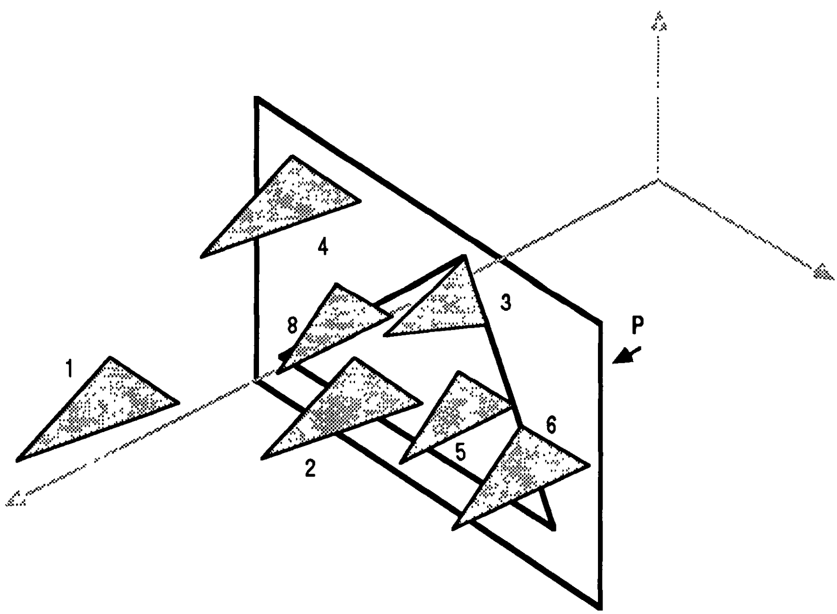



Figure 10 - Relationships formed by the intersection of the boundary set X of a face a2 with the boundary, interior and exterior sets (A,B and C) of a face a1 when the boundary set of of the face intersects the point-set D (or E). a2 is shown shaded, however only the black outline is the boundary set of a2

Since the topological relationships are the same no matter which set D or E on either side of the plane **P** the boundary set of $a_2$ intersects, the combinations are shown in the table with the marker offset between D and E. Note that relationship 7 is not possible between two closed connected simplexes. The other seven relationships are shown spatially in figure 10.

The third group of relationships occurs when the boundary set X of $a_2$ intersects both D and E and hence must intersect the sets A,B and C of $a_1$ at an interval whose boundaries correspond to two points from the boundary set X of $a_2$ and interior corresponds to the interior set Y of $a_2$. The possible combinations between the boundary set X of $a_2$ and the boundary, interior and exterior sets A,B and C of $a_1$ when the boundary set X intersects both D and E as well, are shown in table 3.

| | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|
| Exterior A | | | X | | X | X | X |
| Boundary B | | X | | X | X | | X |
| Interior C | X | | | X | | X | X |
| Above D | X | X | X | X | X | X | X |
| Below E | X | X | X | X | X | X | X |

Table 3: Set Intersections between the boundary of set X of a2 and the exterior (A), boundary (B) and interior(C) sets of a1 in R3 when the boundary of a2 intersects both of the sets D and E.

The spatial interpretations are shown in figure 11. Note that for relationship 10 in column two, the interior set of the face $a_2$ may be used to derive a second possibility. These relationships are marked 10a and 10b in the spatial interpretations of these relationships, shown in figure 11. In addition, relationship 14 is not possible between closed, connected faces.

By examination of all relationships in figures 8, 10 and 11, the number of unique relationships between faces in $R^3$ is fourteen since relationships 1,4 and 11 are particular types of the disjoint relationship shown in figure 8 and relationships 3, 6 and 13 are particular types of the meet relationship shown in figure 8.

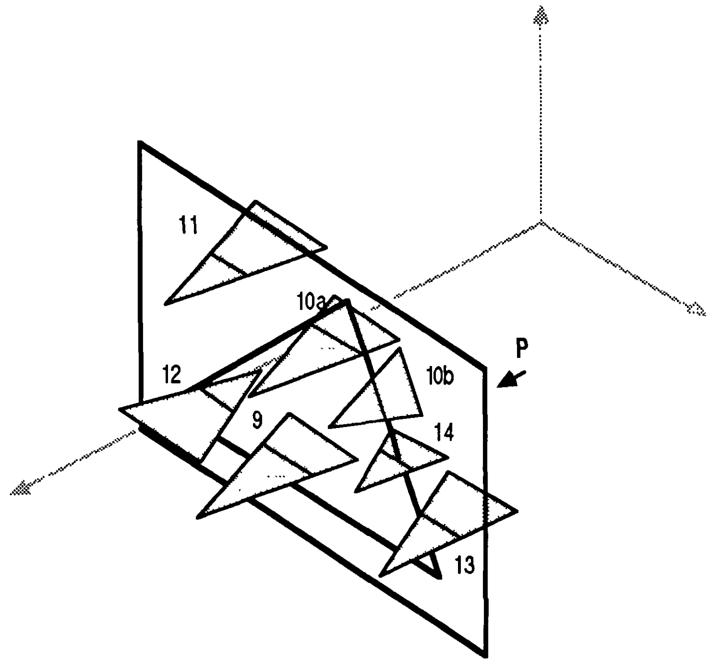



Figure 11 - Relationships formed by the intersection of the boundary set X of a face a2 with the boundary, interior and exterior sets (A,B and C) of a face a1 when the boundary set of of the face intersects both point-sets D and E (passes throught the plane P formed from the union of the boundary, interior and exterior sets of a1. Although a2 is shown shaded, only the black outline is the boundary set

To reduce these fourteen relationships in detail, the union of the boundary and interior points-sets of $a_1$ and $a_2$ in each relationship is considered. Relationships which are homeomorphic can then be reduced to their homeomorphs. Thus, the complete two layer hierarchy of binary topological relationships between faces (2-simplexes) in $R^3$ is shown in figure 12.

Volumes (3-simplexes)

The boundary, interior and exterior sets of a volume (or 3-simplex) $v_1$ in $R^3$ are the same as for a face in $R^2$ (Figure 7). There is a single closed boundary set (B), a single open exterior set (A) and a single open interior set (C) just as there was for faces in $R^2$ in the previous section. The union of sets A,B and C is a 3-manifold equivalent to $R^3$. All possible binary topological relationships between volumes in $R^3$ can then be derived from the possible set relationships between the boundary, interior and exterior

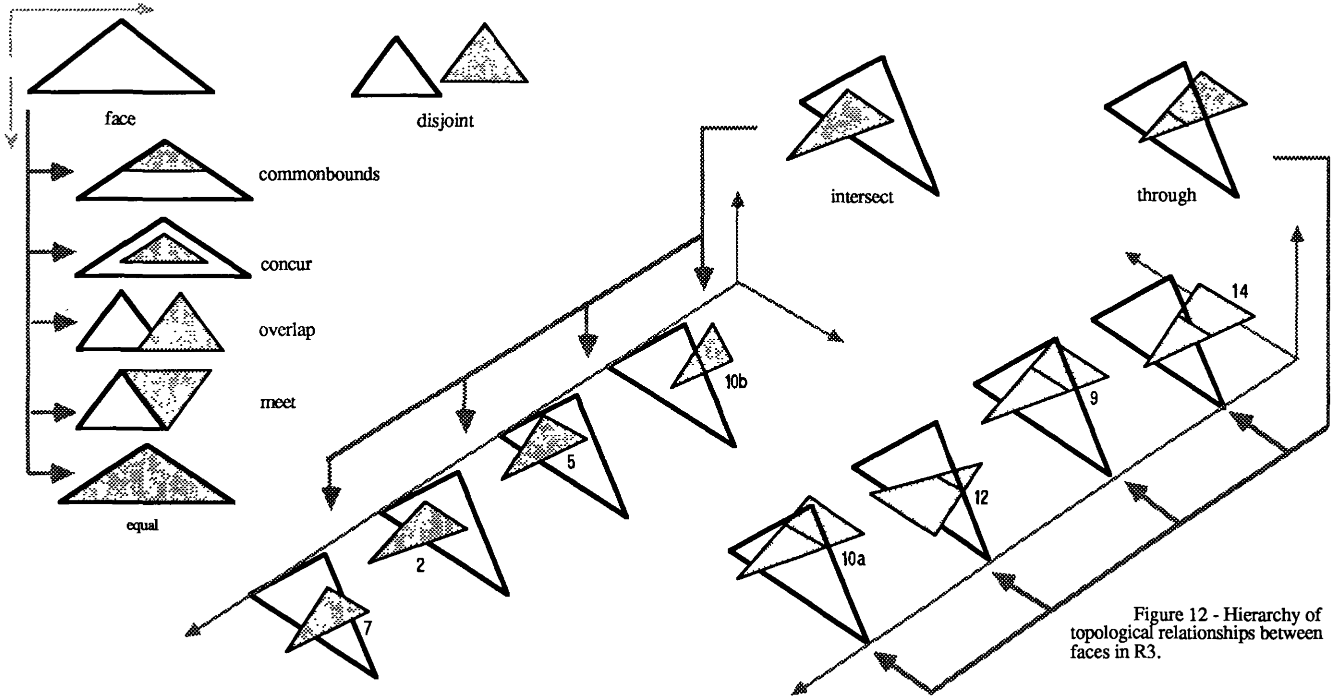



Figure 12 - Hierarchy of topological relationships between faces in R3.

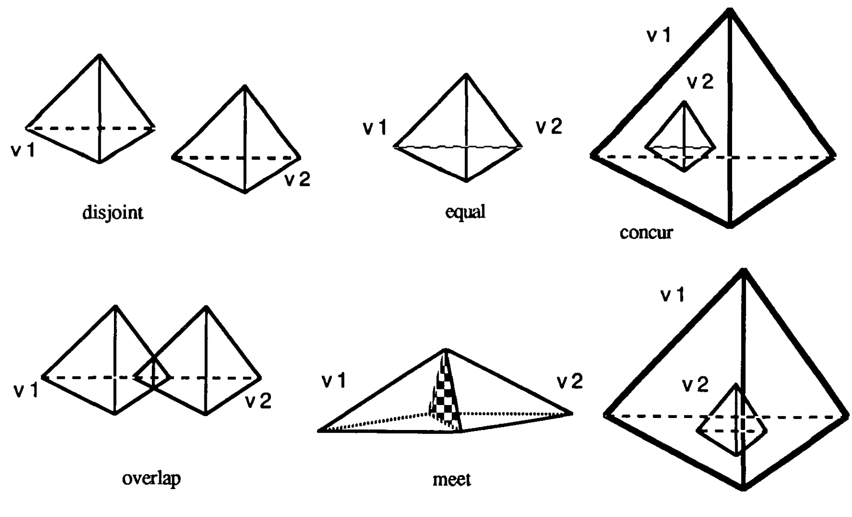
sets A, B and C of $v_1$ and the boundary set X of $v_2$. e.g. if the boundary set X of $v_2$ is contained within the interior set C, then the volume $v_2$ will be contained within $v_1$. The combinations matrix showing the possible relationships between the boundary of a volume $v_2$ and the exterior, boundary and interior sets A,B, and C of $v_1$ is shown in table 4.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Exterior A | x | | | x | | x | x |
| Boundary B | | x | | x | x | x | |
| Interior C | | | x | x | x | | x |

Table 4: Set intersection relationships between the boundary set of v2 and the interior, exterior and boundary sets of v1 in R3

Note that the seventh relationship in the last column of table 4 is not possible in $R^3$ because of the restriction to closed, connected volumes. Not surprisingly the relationships are the same as those between closed, faces in $R^2$.

The six distinct relationships and their names are the same as those used in (Egenhofer et. al. 1990). The spatial interpretations are shown in figure 13.




commonbounds

Figure 13 - Six fundamental relationships between the boundary set of a volume v2 and the boundary, exterior and interior sets of a volume v1

Note that is also possible to use the sixteen different boundary-interior set intersection combinations and the theory shown in (Egenhofer et. al. 1990), to derive the same eight relationships between volumes or 3-simplexes. The only change in the theory required is the use of an extension of the Jordan-Brouwer separation theorem to $R^3$, given in (Alexander 1924).

To reduce these 8 relationships in detail, the union of the boundary and interior sets of $v_1$ and $v_2$ in each relationship is considered. Relationships which are homeomorphic can then be eliminated. For volumes in $R^3$, this results in meet, overlap, contains, equal and common-bounds all homeomorphic to a single volume.

## 5. Conclusions and Future Research

The final aim of this research is a compact and powerful spatial information system for 3D modelling and analysis. Since topological situations in 3-space are complex and difficult, a natural starting place for the development and investigation of a 3D neighborhood topological model is to limit the types of relationships to those that may occur between simplexes since they may be generalized to complex problems via a simplicial decomposition. The topological relationships limiting the cell complex as currently used in 3D topological models for SIS and CAD have been described. To provide a better theoretical basis for 3D situations, a generic and reusable method for deriving fundamental point-set topological relationships between two closed, connected n-simplexes (genus zero) in $R^{n+1}$ (and higher dimensions) has been developed. The generalized method of derivation can be summarised in two steps;

1. Consider the set intersection of the boundary set of a single n-simplex $n_2$ with the boundary, interior and exterior sets of a second n-simplex $n_1$ in $R^n$.
2. Extend these relationships by including either or both of the two additional sets created by embedding the n-manifold created from the union of the boundary, interior and exterior sets of $n_1$ in $R^{n+1}$.

Using this method, the derived sets of binary topological relationships for $R^3$ have been presented as a two-layer hierarchy. Relationships in the first layer are created by considering the union of the boundary and interior sets of the two n-simplexes and replacing those relationships in the second layer which are homeomorphic with a homeomorph. The results are as follows;

| | Second Layer - Fundamental | First Layer - Aggregated |
|---|---|---|
| 1-simplexes in $R^3$ | 8 | 4 |
| 2-simplexes in $R^3$ | 14 | 4 |

| | | |
|---|---|---|
| 3-simplexes in $R^3$ | 6 | 2 |

It is interesting to note that the relationships between 3-simplexes verify the correctness of the extended set of relationships between faces or 2-simplexes in $R^3$. Each relationship between faces or 2-simplexes implied by the eight 3-simplex relationships is predicted within the extended set of 2-simplex face relationships. Similarly, the relationships between 2-simplexes verify the extended set of 1-simplex relationships in $R^3$.

Future research will concentrate on the development of a 3D neighborhood topological model for SIS, the basis for the modelling sufficiency and analytical power of this model will be the relationships derived in this paper. In addition, other hierarchies of these relationships based on set and order theory will be investigated.

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