PASS LOCATION TO FACILITATE THE DIRECT EXTRACTION OF WARNTZ NETWORKS FROM GRID DIGITAL ELEVATION MODELS

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ABSTRACT

The topologic arrangement of peaks, pits, ridges, valleys, passes and pales that form a cartographic surface has been called a Warntz Network. Most network extraction algorithms in the literature focus primarily on either the valleys or the ridges and are not easily applied to extract the full connected Network. An algorithm has been implemented that first locates passes and then traces ridges to peaks and valleys to pits to produce the full Warntz Network from a grid DEM. Three critical point location algorithms from the literature have been tested for their ability to accurately locate passes. In addition, a new algorithm has been developed that starts with the maximum elevation on the surface and then steps through each unique elevation, checking for changes in the eight-connectivity of the regions above and below the current elevation. If a change is detected it is checked to see if it represents a pass. The results from testing the system on several mathematical and terrain surfaces show that the new pass location algorithm performs significantly better than the other three algorithms, producing far fewer anomalous passes and missing only a few passes. The resulting network closely approximates the true Warntz Network.

INTRODUCTION

The study of cartographic surfaces including topography has been an active area in the field of analytical cartography and geographic information systems. Important work has been completed to develop methods for surface representation, visualization and analysis. The fairly recent production of medium scale grid digital elevation models (DEMs) has encouraged developments in computer-assisted geomorphology, geology and hydrology. In particular, many large scale hydrological models have been implemented. In addition to this practical work, a smaller group of researchers have been developing a theoretical basis for cartographic surfaces. In 1966, William Warntz, a key member of this second group, extended some of the ideas put forward by Cayley (1859) and Maxwell (1870) to produce a model of the structure inherent in a cartographic surface. The network he described, composed of ridges, valleys, pits, peaks, passes and pales, has been called a "Warntz Network."

This paper describes a system that can directly extract a Warntz Network from a grid digital elevation model. Since no full algorithm for doing this was located in the literature, an experimental approach has been used to design and test an extraction method formed from a combination of existing algorithms. It is based on an idea presented by Mark (1978) that begins by locating the passes on the surface and then uses them as starting points to define the remainder of the network. To assist in the development and testing of this algorithm, an interactive cartographic system has been built that allows the user to execute a particular combination of pass location algorithms and slope line tracing algorithms to build the network. Both visualization and analysis tools are included in the system to assist in assessing the effectiveness of each combination of algorithms.

CONCEPTUAL BACKGROUND

Before describing the elements of the algorithm in more detail and discussing the results of network extraction, it is useful to review the theory upon which this work is based, including a formal definition of the Warntz Network.

When presented with a terrain surface, the human mind quickly identifies the structural elements. For example, mountains, valleys, ridges, lakes and passes may be remembered as distinct elements of the surface. In the late 19th century Cayley (1859) and Maxwell (1870) described how these features could be defined by the pattern they formed on a contour map, but it was not until a century later that the definitions were pulled together by William Warntz (1966) into a combined surface model. Warntz realized that features on a surface do not occur in isolation but must be interrelated. He described a network composed of peaks, pits, passes and pales as nodes and ridges and courses (valleys) as edges. The ridge and valley lines are slope lines (lines of maximum gradient) that flow from a peak through a pass to a peak or from a pit through a pass to a pit respectively. His definition provides the basic concepts that have been used to develop a structural model for continuous cartographic surfaces.

The Warntz Network model has been modified and clarified by several scholars. A rigorous definition using the language of graph theory is given by Pfaltz (1976). Pfaltz also demonstrates how the model can be used to generalize the graph while maintaining its basic structure. Additional research in the use of graph theoretic procedures to generalize the Pfaltz Graph model of a surface is reported by Wolf (1991). In this work, Wolf shows that the surface can be more accurately generalized by incorporating the elevation change along the arcs in the graph, forming what he calls a metric surface network.

Though the development of a graph theoretical model describes the structure of the network, it is also desirable to have firm definitions for the specific features of which it is composed. Pits, peaks and passes are easily described as critical points of the surface (*i.e.* points at which the magnitude of the surface gradient is zero). Critical points can be further classified as pits, peaks and passes by examining the Hessian matrix of second derivatives. A formal description of all the elements of the surface network, including ridges and valleys, is given by Nackman (1984). He restricts the model to Morse surfaces which are twice differentiable with no degenerate critical points. The surface network (called a two-dimensional critical point configuration graph in the paper) is composed of the critical points of a surface (points for which both x and y partial derivatives are zero) and the flow lines that connect them. An ascending slope line through a point is a vector-valued function of one variable from the real numbers to the real plane defined in the following manner:

$$\bar{\sigma}(t): \Re \rightarrow \Re^2 \quad (1)$$

$$\bar{\sigma}'(t) = \nabla f(\bar{\sigma}(t)) \quad (2)$$

$$\sigma(0) = x_0 \quad (3)$$

A descending slope line is defined similarly but

$$\bar{\sigma}'(t) = -\nabla f(\bar{\sigma}(t)) \quad (4)$$

On a Morse surface every point has a single slope line that runs through it (at critical points this flow line is a single point). A slope line is said to reach a point if the limit of the path as $t \rightarrow \infty$ approaches the point. A ridge is defined to be an ascending slope line from a pass that reaches a second pass or a peak and a valley is defined to be a descending slope line from a pass that reaches a second pass or a pit.

Warntz Networks, or closely related structures, have been studied or used in several different areas of research. Cartography provides the basic theory for spatial surfaces. In hydrology the ridge and channel network structure has a strong similarity to the Warntz Network. In terrain analysis and geomorphology, the Warntz Network has been used to separate study areas into different geomorphic units and has been used to generalize topographic surfaces. In image processing many of the same topographic concepts are applied to identify the "ridges" and "valleys" in a gray scale image. Since the theory and algorithms discussed in this paper were developed within these research areas it is useful to briefly review how each of these areas makes use of surface networks.

The field of cartography has historically been focused on the study of the map as a communication tool. In 1984 the definition of a map was expanded to include 'virtual' maps in additional to the traditional 'real' maps (Moellering, 1984). These virtual maps have either no permanent tangible reality (CRT displays, mental maps, etc.), or are not directly viewable (digitally stored maps, field data, etc.). The field of analytical cartography is based on the analysis of maps using this expanded definition. An immediate benefit gained from this expanded view is the ability to identify 'deep structure' (the structural relationships that can be represented in a virtual map but are not visible in the surface representation of the real map) (Nyerges 1991). It is the deep structure of the DEM (a virtual map) that is captured by the Warntz Network.

Hydrologists use ridge and channel network models as an important tool for studying hydrological processes. A good survey of the many algorithms that have been proposed for extracting drainage networks and basin boundaries from grid DEMs is given by Band (1993). One method uses the local neighborhood to compute the magnitude of the surface gradient and thereby infer surface water flow. The flow direction is assigned towards the neighboring cell with the largest downward slope. Once a unique flow direction has been assigned for each cell, the total number of cells upstream from each cell can be calculated. Channels are determined by selecting cells that drain an area larger than a fixed threshold. The resulting collections of cells are then thinned to single cell-width lines to show the channel centers. A second method uses the morphologic structure of the land to predict the presence of channels. A local operator is used to determine the structure in a neighborhood of the point. If the land is concave upward, the cell is classified as a channel. As with flow line tracing, the cells that are classified as channels will need additional processing. In this case, the networks are not guaranteed to be continuous. To connect the segments of the network, flow lines can be traced downhill from one segment to another to complete the network (Band 1986).

Like hydrologists who are interested in determining the shape and structure of drainage basins, geomorphologists are also interested in the structure of the landscape. In addition to drainage basins, they are interested in the geologic and other morphologic features that exist in the landscape. Several models, including those presented by Dikau (1989) and Feuchtwanger and Blais (1989), have been designed to assist in the storage and analysis of surface structure. Since it is desirable to store a detailed model for use at multiple scales, it is useful to be able to generalize these surface models. A review and conceptual basis for terrain generalization is given by Weibel (1992). For the extended surface network a set of formal generalization rules has been defined based on graph theory. It was shown by Pfaltz (1976) that the Pfaltz Graph can be generalized by contracting a set of peaks and passes into one peak or a set of pits and passes into one pit. Wolf (1991) improved on this work by incorporating the elevation difference into the surface network. In the new structure, the elevation difference and local structure can be used to locate less important elements of the network for contraction.

Many of the concepts developed for terrain representation and analysis have been applied to the interpretation of gray-scale imagery. If one considers the gray levels in an image to represent elevation values, the image can take on many of the characteristics of a terrain surface. One of the first attempts to use surface structure to interpret gray-scale images is described by Toriwaki and Fukumura (1978). The local technique developed by Peucker and Douglas (1975) was applied with a slight modification to separate the image into peak, pit, ridge, ravine, hillside, and pass pixels. The pixels were then thinned and connected into a network much as described by Band (1986). A more complex method for identifying local structure was introduced by Haralick (1983). Instead of directly using the discrete gray-level values, a bicubic surface patch was fit to the 5x5 neighborhood of each pixel. The directional derivatives of this continuous, smooth function were factored to determine the location of zero-crossings, which identify the position of surface specific points. Once these points were located, the sign of the second directional derivative was used to determine if the crossing was a ridge or valley. Pixels containing both ridge and valley crossings were classified as saddle points.

METHOD

An interactive cartographic system has been built to investigate methods for extracting Warntz Networks. The system, which is written in C on a Macintosh personal computer provides 2D and 3D views of the surface and permits the user to control and analyze the network extraction process. Four pass location algorithms were used to generate starting points for the slope line tracing procedure. These were tested on two mathematical and two empirical surfaces.

Three algorithms for locating critical points on a grid DEM were selected from the literature: one from Peucker and Douglas (1975), one from Toriwaki and Fukumura (1978) and one from Haralick et al. (1983). A fourth algorithm has been designed using a significantly different method, but is based on insight gained into the pass location process by studying these three existing algorithms.

The simplest of the three pass-location algorithms is that presented by Peucker and Douglas (1975). It is based on a idea proposed by Greysukh (1967) that determines the local character of a surface by tracing a circle through the neighbors of a point. The number of times the circle crosses the elevation of the center point and the order in which this occurs can be used to characterize the central point. If the point is a pass, it must be a maximum in one direction and a minimum in another. Therefore, the circle must cross the plane of the center at least four times, twice for the maximum and twice for the minimum. The effect of high-frequency noise can be somewhat controlled by requiring the local negative and positive relief to exceed a set threshold. Though the algorithm they describe requires relief in only one direction to exceed the threshold, for this work the algorithm has been slightly modified to require a significant amount of relief in both directions. This modified version of the algorithm will be called the "Peucker/Douglas pass algorithm" for the purpose of this paper.

The second algorithm tested for locating passes is an enhancement of the Peucker and Douglas algorithm and is presented by Toriwaki and Fukumura (1978). Though the emphasis is on the extraction of features from digital images, the technique can easily be applied to DEMs. Two local statistics are calculated for each cell in the image or DEM: a connectivity index (CN) and a curvature index (CC). Since the connectivity index returns the number of connected strings of cells above the central cell, a value of two or above indicates that it is a pass. This algorithm will be called the "Toriwaki/Fukumura pass algorithm" for the purpose of this paper.

The third pass-location algorithm that has been tested also comes from image analysis and is given by Haralick et al. (1983). For each cell a cubic polynomial surface is fitted to its 5x5 neighborhood through the application of a set of ten local filters, one for each term in the polynomial. The Hessian matrix of second derivatives of the polynomial is then computed. If the matrix has two distinct eigenvectors, the roots of the directional derivative in each of these directions are computed to find zero-crossings (if there is only one eigenvector, the Newton direction and the direction orthogonal to the Newton direction is used). If a zero-crossing is found within the area of the cell, the Hessian matrix is recomputed at the crossing point and used to classify the point as a ridge, valley or pass. If a pass or both a valley and a ridge are found in a cell, the cell is marked as a pass. The implementation of this third algorithm will be called the "Haralick pass algorithm" for the purpose of this paper.

The three algorithms described above have one important characteristic in common. They all must make a decision on the character of a cell in the DEM based on the limited amount of information present in a 3x3 or 5x5 neighborhood. This may cause structures with a primary spatial frequency lower than this to be missed. For instance there is often a wide flat area at the center of a pass. If it is large enough, the entire local neighborhood will have equal values and will be interpreted as being flat. The thresholding function found in most image processing systems has been the inspiration for a new pass location algorithm that is able to overcome this problem. If all areas above or below a threshold elevation are grouped and the changes in topology of the groups as the threshold varies are investigated, it is clear that the appearance or disappearance of groups identifies minima and maxima and the connection of groups identifies passes. This is illustrated in Figure 1. If the boundary between groups is viewed as a contour line, the connections correspond to the self-intersecting contours that Cayley, Maxwell and Warntz all used as indicators of the location of a pass.

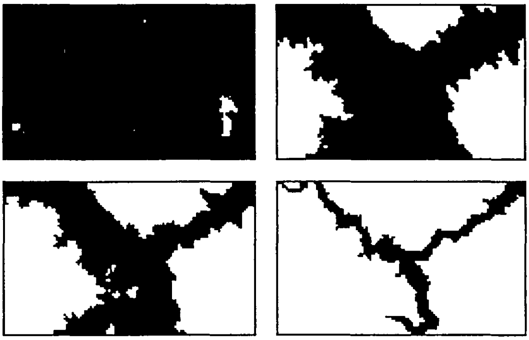


FIGURE 1. Threshold at four different elevations (higher areas are in white)

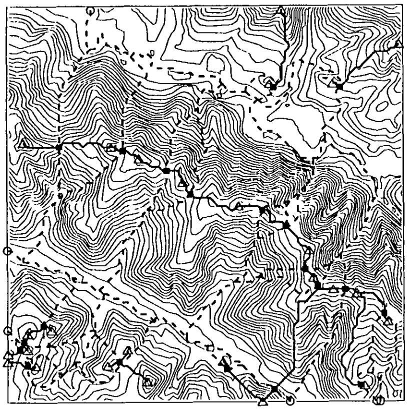
The newly developed algorithm applies this thresholding concept to locate passes. It starts with the set of the cells with the highest elevation. This set is enlarged by systematically adding each layer of lower elevation values to the set until all elevation levels have been processed. As each layer is processed, every cell that is added is checked to see if it changes the eight-connectivity of the higher elevations or the four-connectivity of the lower elevations. If it does change the connectivity, it represents a pass feature; however, some additional checks are necessary to eliminate passes that are too small. This is done by checking that there are no adjacent passes and that there are at least two ridges and two valleys originating from the pass. If the cell passes all of these tests, it is flagged as a pass.

Once the passes are located using one of the four algorithms described above, they are processed by the slope line tracing algorithm which takes a pass location as input and traces the network slope lines emanating from the pass in two steps. The first step is to identify at least two ridge starting points and two valley starting points in the eight neighbors of a cell. The algorithm handles most cases gracefully and forces starting points if necessary (for example in flat areas). Each of these starting points is passed to a second algorithm that attempts to trace a slope line either uphill or downhill starting at the desired neighbor. This second algorithm is complicated by the need to check for intersections and maintain the vector topology as each line is traced. In flat areas, a procedure is used that first identifies the extent of the area and locates any possible outlets. It then builds a distance surface starting at the outlets which is used to direct the slope line to the nearest outlet. In other areas (usually the majority of cases) the slope line is directed to the lowest or highest eight-neighbor using the method described by Jensen and Domingue (1988).

Both artificial and real surfaces have been used to test the system. Two artificial surfaces have been computed from mathematical functions and provide well-defined networks to test the accuracy of the algorithms. The first mathematical surface was generated for a 50 row by 50 column area using a simple cosine function for which the Warntz Network is easily determined. The other mathematical surface was taken from surface III in (Morrison, 1971) and sampled for a 100 row by 100 column area.

The two real surfaces, from California and Colorado, test how well the algorithms perform in a less controlled environment. To provide a model for comparison, a "truth" Warntz Network for each of the surfaces was created by carefully locating the passes on the surface and tracing the slope lines both up and down to capture all the structural elements.

The first terrain data set is a portion of the La Honda California USGS 1:24,000 DEM. This area has been the subject of previous work and therefore its characteristics are well understood. The surface is composed of one main ridge that is flanked by two main



▪ Passes ——— Ridges
△ Peaks - - - - - - - Valleys
○ Pits ---------- Connectors

FIGURE 2. "Truth" network for the La. Honda, California surface.

valleys flowing southeast to northwest. The portion selected for testing and shown in Figure 2 is 100 rows by 100 columns of 30 meter cells starting at row 200 and column 75 in the original DEM. The "truth" network that has been determined by the author is shown with 10 m contours in Figure 2. Note that because some of the ridges and valleys leading into the area and leaving the area originate at passes outside the area, they are not captured in the "truth" network.

The second terrain data set is a portion of the Platte Canyon, Colorado USGS 1:24,000 DEM. It is larger and structurally more complex than the La Honda, California surface and was chosen because it contains well-defined terrain and both the DEM and hydrology layer were easily obtained from the USGS. The portion selected for testing is 154 rows by 253 columns of 30 meter cells starting at row 266 and column 87 in the original DEM.

RESULTS

Each of the four pass location algorithms was tested with each of the four surfaces to generate a total of 16 networks. The four networks for the La Honda, California surface are shown in Figures 3 to 6. A thinning function has been applied to all networks to eliminate adjacent passes and produce a more correct network. The passes located by each pass location algorithm are broken down in Table 1 into the number and percentage of correct and anomalous passes and the number of correct passes that were missed. A clear pattern is present in the data. The Peucker/Douglas and Toriwaki/Fukumura algorithms tend to locate the correct passes, but also add many more anomalous passes. The Haralick and the new pass location algorithms are more conservative and add fewer passes. In addition, the new pass location algorithm correctly identifies most passes, giving it the best of both worlds. The advantage of the new pass location algorithm is particularly evident with the empirical surfaces for which the other algorithms identify three to six times too many passes. This large number of anomalous passes almost completely dwarfs the number of correct passes which only represent 14% to 18% of the total. In contrast, the number of correct passes located by the new pass location algorithm make up a full 61% to 69% of the total.

The Peucker/Douglas pass location algorithm performs fairly well on the mathematical surfaces. However, the accuracy on empirical surfaces is much worse. For each correctly located pass, the algorithm also identifies more than five anomalous passes. This is amplified by the many anomalous slope lines generated from each of the passes. A close inspection of the surfaces has identified three cases that are incorrectly classified by the algorithm. The first case occurs when a wide pass is centered on a flat area that is larger than the 3 x 3 neighborhood used in the algorithm, causing it to be missed. The second case is caused by the indeterminate nature of the cell diagonals. The algorithm assumes that it is possible to connect a cell with its diagonal neighbors, but a ridge or valley structure may prohibit this. The third case occurs along passes that are at least two cells in length. The algorithm makes no provision for checking neighboring cells and therefore identifies both pass cells individually creating a duplicate pass. To some extent, the pass threshold can be adjusted to reduce the number of anomalous passes. However, by requiring the local relief to be greater than the pass threshold, important passes with low spatial frequency, that often have little local relief, might be missed. For the results given in Table 1

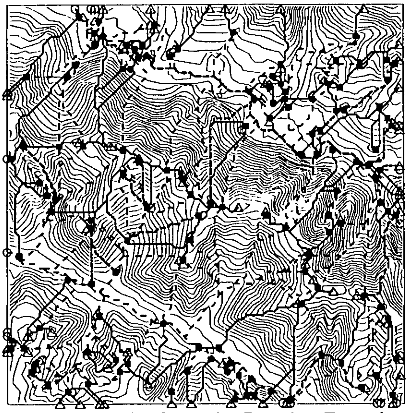


Figure 3. Results from the Peucker/Douglas pass location algorithm

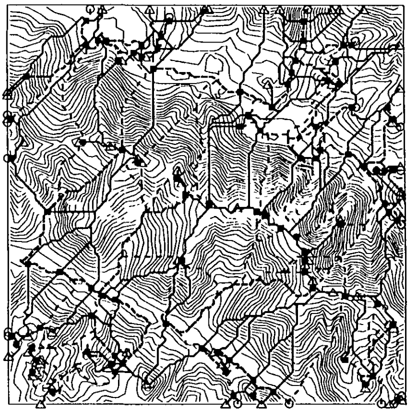


Figure 4. Results from the Toriwaki/ Fukumura pass location algorithm

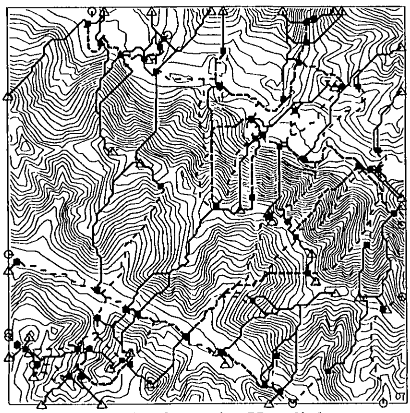


Figure 5. Results from the Haralick pass location algorithm

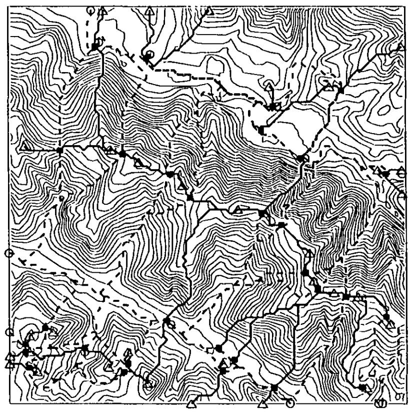


Figure 6. Results from the new pass location algorithm

the pass threshold has been adjusted as high as possible so that no passes are missed. There appears to be no direct method for determining the best threshold value without a prior knowledge of the pass locations.

The Toriwaki/Fukumura pass algorithm has many of the same attributes as the Peucker/ Douglas pass algorithm described above. It is also restricted to a 3 x 3 neighborhood and makes the same assumptions about the position of ridges and valleys that cut across diagonals. Though no threshold is used, fewer erroneous passes are identified. Unfortunately, it also tends to miss more passes. One particular problem is evident in the processing of the double cosine surface. Only one of the five passes is found due to the way equal-value neighbors are treated. Since the algorithm is designed to operate on boolean values, the neighbors must be classified as either above or below the center cell. Therefore, an arbitrary choice must be made to classify equal-value neighbors as either above or below. This choice can cause passes to be missed in even very small flat areas.

The Haralick pass algorithm benefits from its larger 5 x 5 neighborhood that permits it to identify pass structures that are too broad to be captured by a 3 x 3 neighborhood. It is also less sensitive to local variations due to the smoothing effect of the polynomial surface patch. The passes identified by the Haralick algorithm tend to be accurate. However, it is

TABLE 1. Pass Location Algorithm Results

Double Cosine (5 passes)	Total	Correct*		Anomalous*		Missed**	
Peucker/Douglas	5	5	100%	0	0%	0	0%
Toriwaki/Fukumura	1	1	100%	0	0%	4	80%
Haralick	5	5	100%	0	0%	0	0%
New algorithm	5	5	100%	0	0%	0	0%
Morrison III (33 passes)	Total	Correct*		Anomalous*		Missed**	
Peucker/Douglas	35	33	94%	2	6%	0	0%
Toriwaki/Fukumura	39	33	85%	6	15%	0	0%
Haralick	27	27	100%	0	0%	6	18%
New algorithm	33	33	100%	0	0%	0	0%
La Honda, California (20 passes)	Total	Correct*		Anomalous*		Missed**	
Peucker/Douglas	134	19	14%	115	86%	1	5%
Toriwaki/Fukumura	96	15	16%	81	84%	5	25%
Haralick	57	8	14%	49	86%	12	60%
New algorithm	31	19	61%	12	39%	1	5%
Platte Canyon, Col. (88 passes)	Total	Correct*		Anomalous*		Missed**	
Peucker/Douglas	581	85	15%	496	85%	3	3%
Toriwaki/Fukumura	371	68	18%	303	82%	20	23%
Haralick	245	44	18%	201	82%	44	50%
New algorithm	125	86	69%	39	31%	2	2%

* The correct and anomalous passes are also given as a percentage of the total passes
** The missed passes are also given as a percentage of the total correct passes.

often too restrictive and does not identify some of the correct passes. These missed passes cause sections of the Warntz network to also be missed, leaving a disconnected network. In the La Honda surface the algorithm missed more than half of the passes and added 49 incorrect passes, significantly affecting the quality of the final network (see Table 1 and Figure 5). The algorithm also suffers from the lack of continuity between surface patches. This can cause a slight inaccuracy in the surface fit to place the critical point outside the pass cell. If this occurs in all cells in the neighborhood of a pass, the pass is missed.

The new pass location algorithm has been devised to address some of the difficulties faced by the other three algorithms. As shown in Table 1, it performs very well on all the surfaces. It identifies as many or more of the correct passes than the other algorithms and does not create nearly as many anomalous passes. This is made particularly clear if the number of correct passes as a percentage of the total passes identified by the new pass location algorithm is compared to the same percentage for the other algorithms on the two empirical surfaces. For the La Honda, California surface 61% of the passes identified by the new pass location algorithm were correct, far more than the 14% to 16% for the other algorithms. Similarly, 69% of the passes identified by the Wilcox algorithm for the Platte Canyon, Colorado surface are correct compared to the 15% to 18% for the other algorithms. The networks produced from the new pass location algorithm passes are very close to the "truth" networks, deviating only in small details that do not significantly affect the structure.

CONCLUSIONS

Though a full Warntz Network extraction procedure is presented in this paper, the focus has been on the pass location component. Early testing of the system indicated that the correct location of passes is the most critical component of the extraction procedure because the entire network must be based on the identified passes. Testing of the Peucker/Douglas, Toriwaki/Fukumura and Haralick pass location algorithms revealed a tendency to miss broad pass features. The algorithms were also shown to be quite easily confused and can recognize small irregularities in the surface as passes. This has led to the conclusion that a pass location algorithm must consider the global structure of the surface.

The new pass location algorithm was designed with this goal in mind. By considering the structure of the surface at each unique z-level, the algorithm is able to capture the changes in surface topology that indicate the location of a pass. Since the entire surface is processed at each step, even large flat areas can be recognized as being the center of a pass. The advantage of this approach over using an attribute of the cell's local neighborhood is clearly demonstrated by the significantly larger percentage (61% - 69%) of correct passes reported in Table 1, and shows it to provide a better solution than the other three algorithms for which the correct passes form only a small percentage (14% - 18%) of the identified passes.

The combined Warntz Network extraction system, which starts by using the new pass location algorithm to locate passes, then locates ridge and valley starting points and finally uses the deterministic Jensen & Domingue slope line tracing method to complete the network, has been shown to be a reasonably effective tool for the extraction of Warntz Networks. The networks that have been produced by this method closely approximate the true Network and do not contain the many erroneous elements added by the other combination of algorithms that have been tested.

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