MEDIAL AXIS GENERALISATION OF HYDROLOGY NETWORKS

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Abstract

We examine some benefits of using the medial axis as a centreline for rivers and lakes. One obvious benefit, automatic centreline generation, has been used for many years. We look at how the topological relationships between the medial axis and the river banks or lake shores can provide extra network characteristics such as river areas and opposite river banks. We also report on our experience at approximating the medial axis with a Voronoi diagram of point sites.

1 Introduction

Maps are rich in geometric structure. Adjacencies between features, containment in regions, intersections of lines or regions, and relative orientations or proximities of features all contribute to the value of maps and map operations. For digital maps, much of this structure is lost to computers; the visual cues that people use to see the geometric structure are not available to a computer. Instead, we develop algorithms for topology building, polygon containment, and polygon intersection to capture this structure for map analysis by computers.

Structure, in particular locality and proximity, appears in various forms. Regular grids [21] and quad trees [12, 19] localise points into a small region of space. The medial axis describes the "shape" of polygons in a variety of fields such as map labelling [2], shape matching [13, 15], solid modelling [24], mesh generation [9], and pocket machining [10]. Voronoi diagrams [3, 8] capture both the locality of objects as well as their proximity to one another for applications such as identifying polygon closures and line intersections while digitising from maps [7]. In this paper, we focus on the medial axis.

Centrelines have been a standard tool of manual cartography for generalising networks for many years. Digital cartography inherits centrelines for generalisation of river and road systems [14], for simplifying the analysis of these same systems, and for extracting linear features from raster models [17]. The medial axis is one method that practitioners have used to generate these centrelines automatically.

The characteristics that make the medial axis a good choice for centrelines can be taken one step further for river networks. An edge of a river's the medial axis is the bisector of the two nearest river banks. Therefore, each medial edge identifies a pair of river banks that are nearest to one another. This nearness relationship allows us to

- associate opposite banks of a river.
- tie analysis on centreline networks to original river bank data.
- calculate surface areas for rivers and river segments.
- extend network orderings, such as the Strahler [22] or the Horton [11] orders on river networks, to include lakes and wide rivers for cartographic generalisation.

Although the medial axis is a well-defined structure, calculating the structure in the presence of degeneracies can be difficult. Consequently, we use an approximation to the medial axis in our experiments. The approximation is based on a robust implementation of the Voronoi diagram for points.

In section 2 we describe our motivation for looking at centrelines of river networks. Section 3 provides some basic geometric definitions. Section 4 gives a few more details on the benefits of the medial axis as a centreline. Finally, our approximation to the medial axis by a Voronoi diagram of points appears in section 5.

The work in this paper has been done in conjunction with Facet Decision Systems.

2 The Problem

River slope, shore length, and surface area are three characteristics that influence the suitability of a river for salmon spawning. In digital maps, rivers appear as a single-line or as a set of river banks. In the first case, river slope and shore length come directly from the single-line rivers and a surface area estimate comes from some nominal width for the river. In the second case, the river is defined implicitly by its banks. We can compute slope and length for individual river banks, but there is no correlation between opposite banks. As with single-line rivers, a nominal width for the river generates an approximation to the surface area, but the approach ignores the implicit information of the map, namely the delineation of the river itself.

River centrelines lead to a better estimate of both length and area for wide rivers. Centrelines are a common approach to converting hydrology networks into tree-like river networks [14]. The length of the centreline averages-out the difference in length of the two river banks as the river meanders. Moreover, a centreline with flow-directed edges establishes upstream and downstream relationships between tributaries on opposite sides of the rivers.

The area of a wide river is a bit more elusive than the length. Although river banks may be labelled as either a right or left bank, digital maps do not usually encode which portion of a river bank is opposite another bank. Consequently, we know where the river is, to the left or right of an edge, but we don't know how wide the river is. The key to getting a better area estimate beyond using a nominal river width lies with finding the centrelines automatically and with making better use of the rivers' centrelines.

In our system, river centrelines are a subset of the medial axis of the river polygons. Efficient algorithms [1, 6] can compute the medial axis of a river automatically and generate river slopes and lengths. As a by-product of the algorithm, each edge of the medial axis is tagged with the two closest river banks and each river banks is tagged with its nearest medial edge. Consequently, given a point on a river bank, we can find the distance from this point to its nearest medial axis edge; this distance is half of the river's width at that point. Knowing the width of the river at any point of our choosing allows us to make a better estimate of a river's surface area.

3 Definitions

The *medial axis* of a polygon [4, 18] consists of the centre of all the circles contained inside the polygon that touch two or more different polygon edges. Polygon vertices, where two edges meet, are counted as a single edge. For any such circle, its centre is equidistant to the two edges that it touches and is therefore on the *bisector* of the two edges (figure 1).

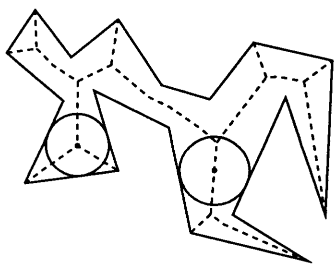


Figure 1: Medial axis of a polygon (dotted) and two touching circles.

The Voronoi diagram [3, 16] of a set of point *sites* is a partition of the plane into maximally-connected regions in which all points share the same nearest sites. Points that are equidistant to two nearest sites form the boundaries of the partition's cells. The cell boundaries, called *Voronoi edges*, are bisectors between the closest sites. Points that are equidistant to three or more nearest sites are called *Voronoi vertices*. In non-degenerate cases, the Voronoi vertices are defined by three sites; if you join the three sites that define a Voronoi vertex then you obtain a *Delaunay triangle*. The same definition of a Voronoi diagram holds when line segments, such as the edges of a polygon, or arcs are the sites instead of points.

4 Medial Axis Benefits

The key aspect of the medial axis in Section 2 is the association between the edges of the medial axis and the nearest banks of the river. This association provides more benefits than simply an estimate of river widths.

First, the association of the medial axis edges to their nearest river banks tie future calculations on the centreline to original data elements. The river centrelines replace the river banks in a network to give a single-line river network. Further analysis, such as identifying drainage basins, locating fish spawning habitat, and tracking the run-off of forest cut blocks, uses the single-line network. Attributes of the medial axis edges from these operations are propagated to the appropriate river banks. The single-line network allows for simpler network analysis without sacrificing links to the river banks.

Second, the medial axis edges define opposite banks. Since a medial axis edge is the bisector of its closest river banks, these two banks are, in effect, opposite one another along the river. Attributes of opposite banks such as slope, elevation, soil type, and vegetation type can then be compared, to detect either inconsistencies of the data or anomalies in the environment.

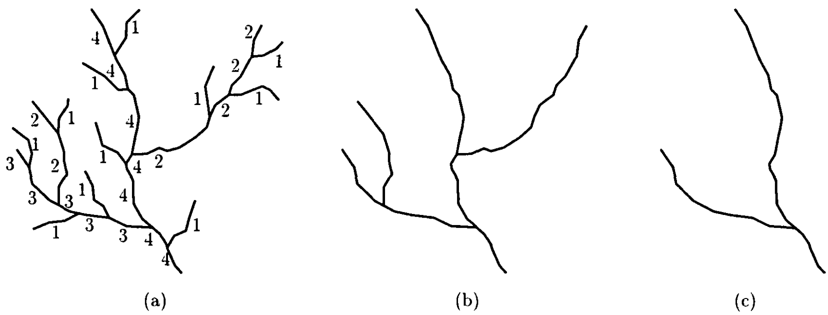


Figure 2: A river network with its Horton order (a) and the subnetworks with Horton numbers greater than 1 (b) and 2 (c).

Third, the association extends network orderings to wide rivers and lakes for cartographic generalisation. The medial axis or centrelines of rivers have long been used to generalise rivers [14]. Network orders on single-line networks, such as the Horton [11], Strahler [22], and Shreve [20] orders, extract the primary branches of a river network for generalisation at large map scales. Figure 2 shows a sample network with its Horton order and the result of selecting edges of only high order from the network. These same orderings can be applied to networks that contain lakes or river banks by treating the lakes and wide rivers as their medial axis; this is not surprising. The original lake shore edges receive an order number from the nearby medial edges. Selecting edges with high network orders will also extract the lakes along the path. A minimum area for the lake is of added benefit at large map scales.

The propagation of network orders to lake shores and river banks can take one of two forms. Lake shores receive the network order of the nearest medial edge or lake shores receive the network order of the highest medial edge in the lake. The latter form treats a lake as a single unit to preserve the visual cues of lake extent and shape and is our preference for network simplifications. (Not all edges of a centreline have the same network order number.)

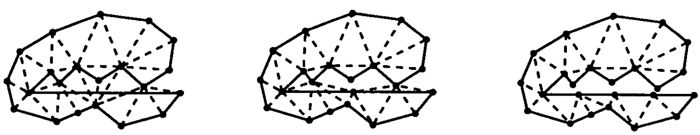
5 Computation of the Medial Axis

The computation of the medial axis is well-studied in computational geometry. Optimal algorithms [1, 6] have been published to compute the structure for simple polygons. The medial axis is also a subset of the Voronoi diagram of the polygon's edge and, as such, algorithms that compute Voronoi diagrams [5, 16, 18, 25] are applicable to finding the medial axis.

Unfortunately, few implementations for computing the medial axis of a polygon available are robust. Although many Voronoi diagram algorithm implementations exist, most do not handle the "degeneracies" of lines that share common endpoints, which arise when you try to compute the medial axis as a subset of a Voronoi diagram. Consequently, we use a robust sweep algorithm for the Voronoi diagram of point sites to approximate the medial axis of a polygon.

5.1 Medial Axis Approximation

Given the polygonal contour of a river or lake, we discretise the boundary of the river, compute the Voronoi diagram of these points, and approximate the medial axis from the result. Theoretically, as more points are added to discretise the boundary of the polygon, the Voronoi diagram inside the polygon converges to a superset of the polygon's medial axis. Computationally, adding more points to the boundary adds degeneracies and increases the computation time. We need to strike a balance between computation time and diagram fidelity.



(a) (b) (c)

Figure 3: Delaunay triangles inside the polygon after (a) the initial point, (b) one decomposition step, and (c) two decomposition steps.

Our solution adaptively discretises the river boundary and starts with the points in the river bank's polygonal lines (figure 3(a)). After computing the Voronoi diagram of these points, we compare each Delaunay triangle of the Voronoi diagram with the boundary of the river: if some river boundary cuts through any Delaunay triangle then we split the edge at its midpoint, add the midpoint to the set of point sites and recompute the Voronoi diagram (figure 3(b)). The result of these iterations is a decomposition of the river's interior into Delaunay triangles (figures 3(c) and 4).

We do not want the entire medial axis as the centreline of a river. We only want those edges that lead to tributaries or that link tributaries; we consider the outflow end of a river to be a tributary. Consequently, we mark all of the Delaunay triangles that have a tributary at one of its vertices and mark the Delaunay triangles inside the river that form a path between the tributaries' triangles.

There are two ways to approximate the medial axis from the marked triangulation. The first uses the subset of the Voronoi diagram whose vertices correspond to marked Delaunay triangles. If the initial discretisation of the river edges resulted in a good triangulation and the points along an edge were far apart from one another then this approximation can look like a zig-zag pattern rather than an expected smooth centreline.

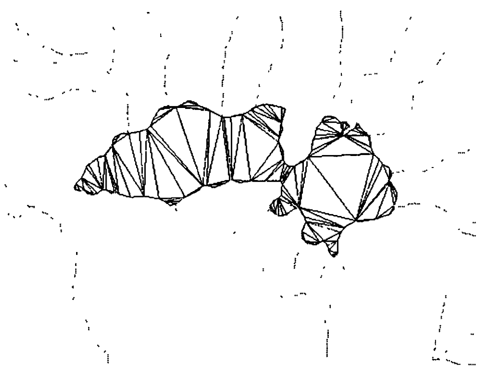


Figure 4: Delaunay triangles inside of a lake.

The second uses a representative point inside each Delaunay triangle and joins the points of adjacent marked triangles into paths. Of course, the result of this method depends on the choice of representative points. One possibility uses the centroid of each triangle. If the base of the triangles alternates between river banks and the triangles have one side much smaller than the other two sides, then the approximation is jagged. Another possibility uses the midpoint of the line between the middle of the shortest triangle edge and its opposite triangle vertex as a representative point and produces a much smoother effect for long and thin triangles. In both cases, the approximation has a tree structure and both the Voronoi edges and the Delaunay triangles record the closest river bank edges.

The medial axis itself does not address the entire problem; the direction of water flow along the medial axis edges has been ignored so far. A correct direction of flow is important when you want to answer queries such as "What is the river area upstream from a particular point?" or "If a particular tributary is polluted, what (downstream) fish spawning habitats may be affected?" In most cases for rivers, the direction of flow for the medial axis edges matches either the flow direction of a tributary at one end of the edge or the flow direction on the river banks that define the medial edge. This does not apply to medial axis edges inside lakes since the lake shore edges do not contain any flow. For lakes, a few simple topological rules have been sufficient in our experiments:

- if an edge is adjacent to a tributary then the flow of the edge matches the flow of the tributary.
- if the medial axis edges inside a lake meet at a node then there must always be at least one edge that enters and at least one edge that leaves the node.

We set the direction of flow on medial edges that lead directly to tributaries according to the first rule before examining the inner edges of the medial axis.

5.2 Area Generation

As mentioned in Section 2, the area of a river can be derived from the river's length and width. When Voronoi edges approximate the medial axis, width measurements are the distance between Voronoi edges and river banks. When paths between Delaunay triangles approximate the medial axis, we obtain the area in a different manner: assign the area of each Delaunay triangle to its representative point. Since the Delaunay triangulation decomposes the interior of the river, the river area between two points is the sum of the areas at the approximation's nodes between the two points.

Computing river area from Delaunay triangles has some drawbacks. Not every Delaunay triangle has its representative point in the approximate medial axis since the approximation only keeps the portions of the medial axis that link tributaries. As seen in figure 5, the areas of some inlets and bays must be allocated to a nearby representative points to preserve all of the feature's area. The variation in granularity of the triangle areas is another drawback. The area of a triangle in a river branch may be small while the area of a triangle at a river junction may be large.

6 Sample Centrelines

Our data is supplied by the Canadian Department of Fisheries and Oceans and Facet Decision Systems. It is a set of coded polygonal lines that outline terrain features. We use the hydrological features: rivers, river banks, and lake shores. The data is grouped in 1:20 000 scale map sheets with a 1 metre accuracy in the xy-plane and a 5 metre accuracy in elevation. The data adheres to the 1:20 000 TRIM data standard of British Columbia [23]: rivers and river banks are digitised in a downstream direction while lake shores are digitised in a clockwise direction. Rivers whose width is less than 20 metres are digitised as the centreline of the river. Rivers whose width exceeds 20 metres have their left and right banks digitised; no association between opposite banks appears in the source data. Although the polygonal lines are not guaranteed to appear in any particular order, the digitising standard mandates two characteristics: polygonal lines only meet at their endpoints, which are numerically identical.

Since the polygonal lines are unordered, we must build the topology of the data before computing the medial axis of the features. Adjacent lines share numerically identical endpoints so we place all the line endpoints into two-dimensional buckets and then use matching points within each bucket to find adjoining edges. The matched ends provide enough topology to trace the outline of lakes and rivers.

We have extracted the directed centrelines and areas of features in the mountainous interior of British Columbia where lakes have few out-flowing rivers. In

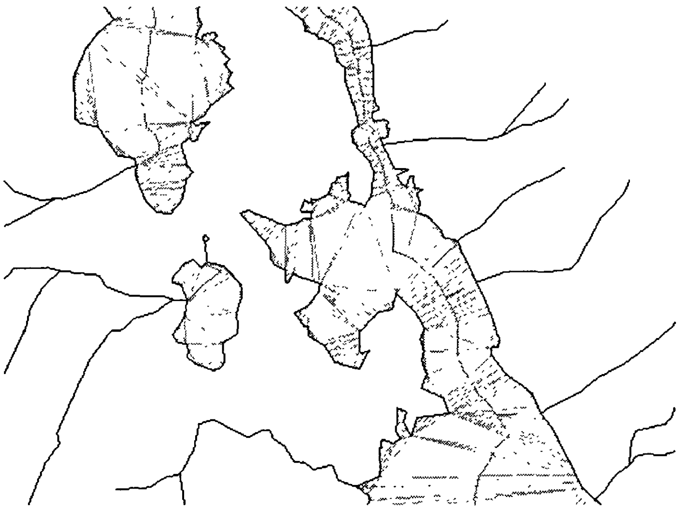


Figure 5: The area of inlets and bays must be assigned to nearby medial edges or vertices.

the 500 lakes and rivers tested, the resulting water flow has been consistent with the expected flow in all of the cases. In the majority of the cases, the lakes only had one outlet and one inlet so deriving the direction of flow is simple. Other rivers or lakes, as in figure 6, have a medial axis that branches more than off just one centreline where the outlets were grouped at one end of the lake. The grouping makes the general water flow patterns simpler and more predictable than a lake with widely distributed outlets.

Although we obtained area estimates for the lakes and rivers in the tested watersheds, the process was not without difficulties. While the data digitising standard seemed ideal for geometric computations, we still needed to find and remove inconsistencies— primarily digitising errors: open polygons, miscoded edges, reversed edges, and missing edges.

Another difficulty, which we have not yet resolved, is the over-estimate of the area caused by islands and sandbars. Sandbars appear along river banks and narrow the effective width of the river. Islands eliminate area from the river. We expect to handle sandbars by using a more liberal definition of a river bank. As for islands, we can subtract their area from the rivers or lakes to which they belong, but this solution is not very satisfactory; it does not give us an easy method for finding the area of a river between two points on the river banks, and the automatically-generated centrelines do not necessarily respect the land formations (figure 6).

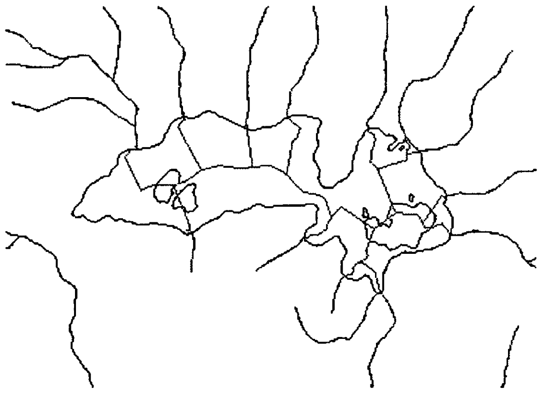


Figure 6: The medial axis of the boundary does not respect islands.

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