MATHEMATICAL MAP MODELS

W.R. Tobler
University of Michigan

Since earliest times the cartographer has used mathematical models to deal with the geometrical content of maps. Most of what one learns in surveying courses bears on this aspect of cartography. The mathematical nature of the model becomes very explicit in analytical photogrammetry, or in the hypotheses about nature which are invoked for computer interpolation in a digital terrain model. Similarly, one has only to consider the reductions which take place after distances or directions have been measured. We can assume that plane Euclidean geometry holds, or that the earth is a sphere, or an ellipsoid; or we can invoke the more recent model proposed by Hotine.[2]

A single example suffices to illustrate some of these points. Figure 1 shows an outline map of the United States. Its construction adheres to the following algorithm, all done using a computer/plotter combination:

(1) Using spherical formulae compute a table of distances between places of known latitude and longitude.

Figure 1: An Empirical Map Projection of the United States.

(2) From this table compute plane coordinates such that separations subsequently computed from these coordinates (by plane formulae) agree, as nearly as possible, with the distances in the original table.
(3) Use the resulting table of latitude/longitude to x/y coordinate correspondences and interpolate detail to complete the final map.

That this algorithm results in a reasonable drawing is clear from the figure; greater mathematical detail is available elsewhere. 9/

Now go to a road atlas and look up the latitude and longitude of all of the places cited in the table of road distances. Operate on this table of road distances with steps (2) and (3) of the foregoing algorithm. The resulting map might appear as in Figure 2. The same algorithm will work with travel times, or travel costs, etc. It comes as somewhat of a shock to the cartographer to observe that the model also works if one knows only ordinal distances (far, farther, farthest), or even only adjacencies, but this simply illustrates the power of topology.4/ In these strange maps we are adopting an alternate geometrical model. Consider the set of all places which you could reach within one hour of travel. Think of this as a geographical circle of one hour radius. Shade it on a map. Is πr^2 the formula for the area? Does your circle have holes in it? Disjoint pieces? The geometry of geography is more complicated than the cartographers might have us believe. Cartograms

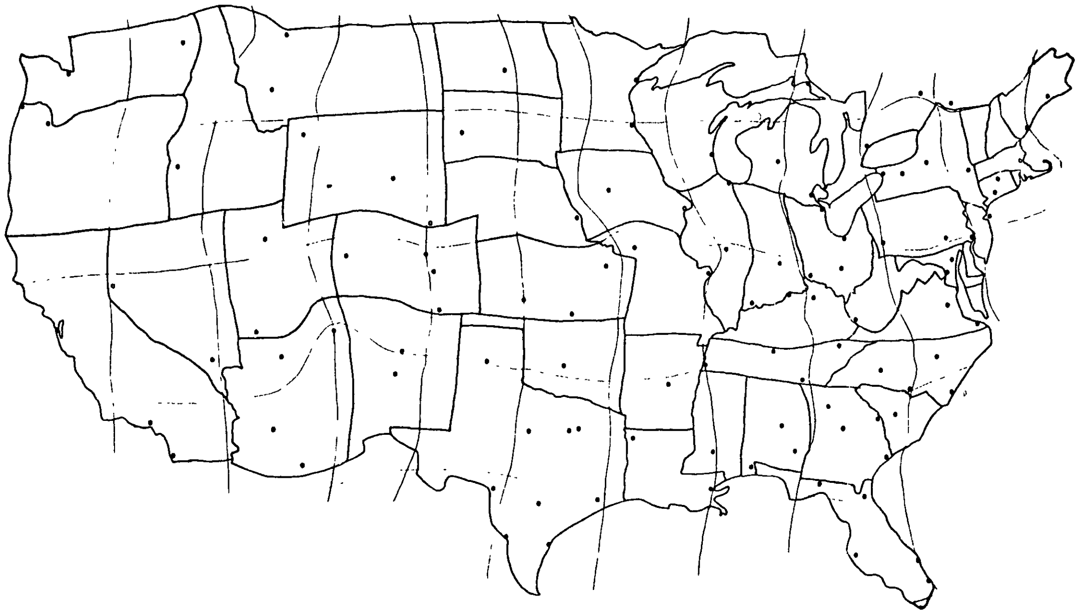

Figure 2: A Road Distance Projection of the United States. By J. Cwirko.

illustrate this from another point of view. 10/ Figure 3 tries to show a New Yorker's map of the United States assuming that the usual mean-information-field measures the amount of detail available to a person living in that city. 11/

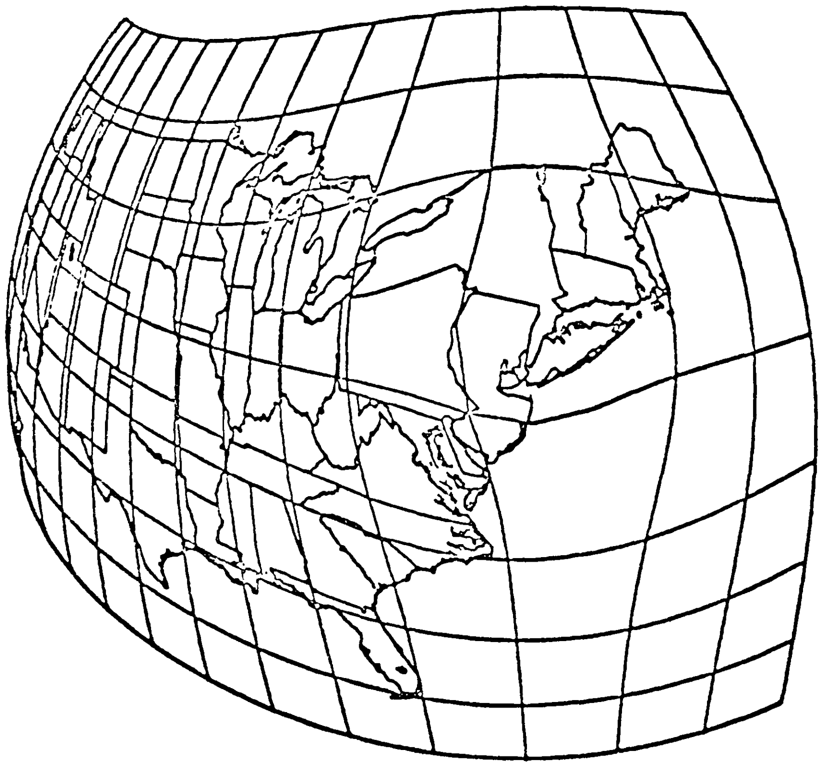

Figure 3: The New Yorker's Mean-Information Field

The substantive content of maps is much less well understood than the geometric content. Frequently the symbols are classified as denoting points, lines, or areas. More recently the measurement level of the data depicted has been cited. This yields the nominal-, ordinal-, interval-, or ratio-data classification. For some purposes I find it useful to consider how many observations apply to each point on the map, further broken down as to whether these are numerical or categorical observations. A presence-or-absence map for one class of phenomena can be thought of as a binary map. A land use map of N exclusive and exhaustive categories is an N-ary map with one observation at each location. A scalar map consists of one real number at each location. This number might represent elevation (as on a topographic map), or population density, etc. On a map of winds one has two real numbers at each location, the components of the wind. A colored map can be thought of as a representation of three distinct positive numbers, the tristimulus values. A map of wind roses really tries to represent an infinite number of numbers at every single point, obviously a difficult task.

The scalar field is clearly one of the simplest things imaginable. I concentrate on it for my second example of a mathematical map model. Scalar fields are often written in the cartographic literature as $z=f(x,y)$; are shown by contour maps, or by computer block diagrams, as in Figures 4, 5, and 6.

Figure 6 is perhaps the most important, for the present purpose, of the three illustrations. It is an unconventional view of an aerial photograph, with the darkness of the grey tones shown as "elevations." The rough topography is a residential subdivision; the irregular trough is a freeway; and the flat area is an empty field. The basic point is now very simple. We can apply to the data of Figures 4 and 5 -- a piece of topography and a statistical "surface" -- the same

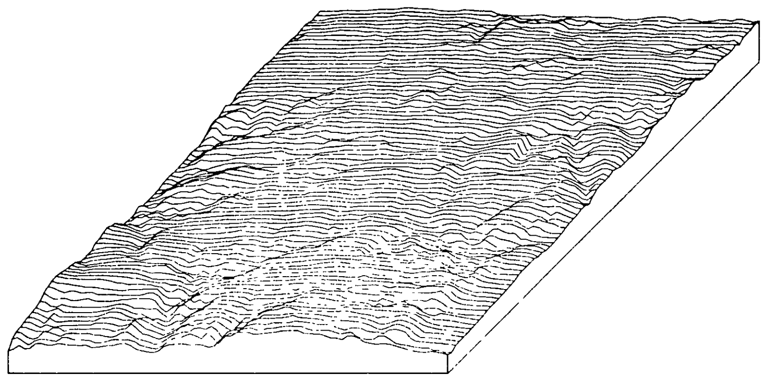

Figure 4: A *Topographic Surface,* $z=f(x,y)$

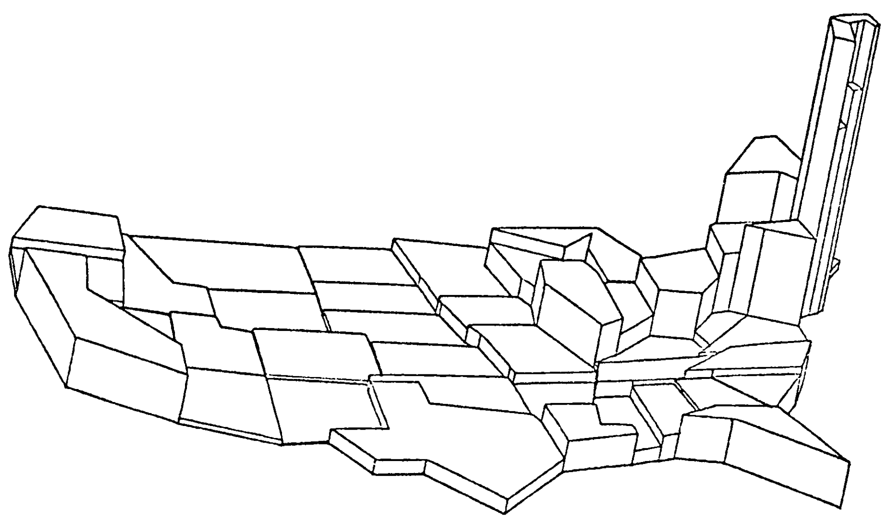

Figure 5: A *Statistical Surface,* $z=f(x,y)$

types of mathematical operation as are applied to photographs. 1/ 7/ The idea is not new 12/, but the applications to date have been rare. For example, one of the common operations on pictures has edge detection as its objective. Mathematically,

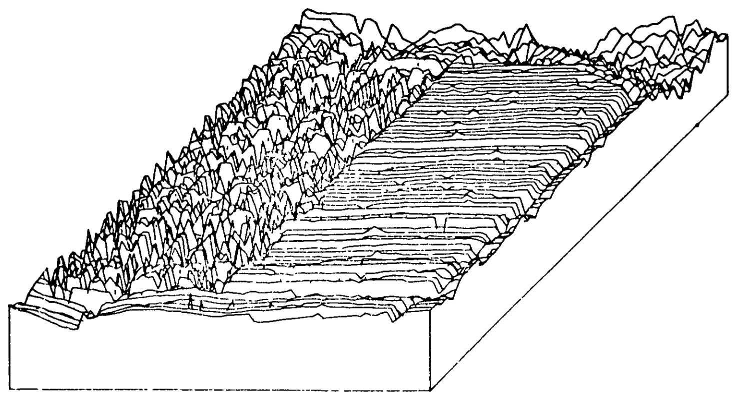

Figure 6: An Aerial Photograph, z=f(x,y)

this consists of taking the second spatial derivative or LaPlacian. Could not this be used to detect the edges of geographical clusters of people, when the input data are population densities? 6/ Another common application in picture processing is noise removal or image enhancement. 3/ There is a nearly one-to-one correspondence between these ideas and map generalization, and attempts to make maps more legible.8/ A basic mathematical idea behind much of this work is that any picture can be considered as the weighted sum of several simpler pictures. Depending on the purpose one may wish to emphasize certain of these basic pictures at the expense of others. 13/

One of the fundamental concerns in the interpretation of aerial photographs is the spatial resolution. This should also be of concern when making statistical maps. Figure 7 shows Holland at five different levels of spatial resolutions. One knows, from the sampling theorem, that phenomena of a size less than twice the sampling interval can generally not be detected. Thus it is very important to choose the spatial resolution on the basis of the problem one is trying to solve.5/ A

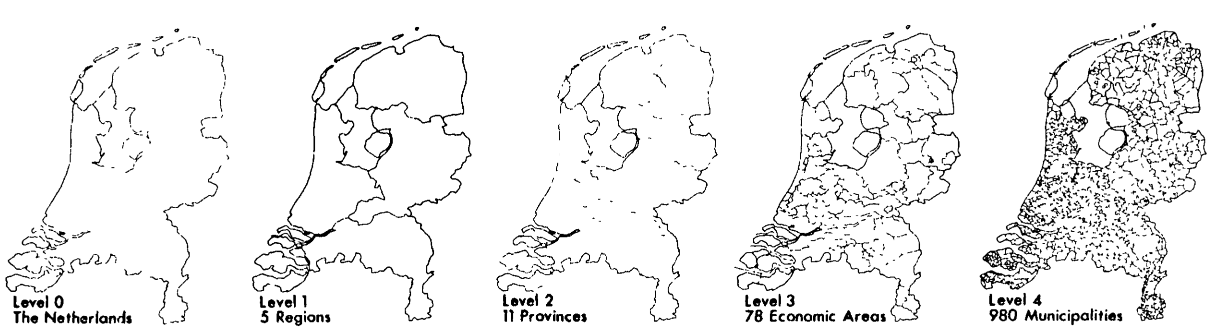

Figure 7: Spatial Resolutions of 200 km, 90 km, 60 km, 23 km, and 6.5 kilometers.

county map of the United States has a resolution of approximately 50 kilometers; phenomena with a magnitude of less than 100 kilometers cannot be detected. State data has a mean resolution of 430 kilometers; features larger than 860 kilometers on a side can be detected.

Digital pictures are also quantized into discrete grey levels, usually measured in bits. A two-bit map would contain four levels of shading. The choropleth map of the cartographer can be viewed as the process of converting the data of Figure 6 back into an image of greys using zip-a-tone or some other shading technique. The fineness of detail in the resulting image is dependent on this quantization process. 14/

As a final example of a mathematical map model consider the case in which there are N numbers at each of N locations; we may refer to this as a vector valued data set. A simple example is a migration table in which both the row headings and the column headings represent the places, and the entries in the body of the table detail the movement of the population during some period of time. The cartographer is faced with the problem of representing all of these data on a map. One solution is to sum across the rows and columns, then take their difference, obtaining the net movement with some places having net immigration, others emigration. The effect is to reduce the N^2 numbers to N, and these can then be shown as a scalar function. But the data could also be considered as arrows flowing from each

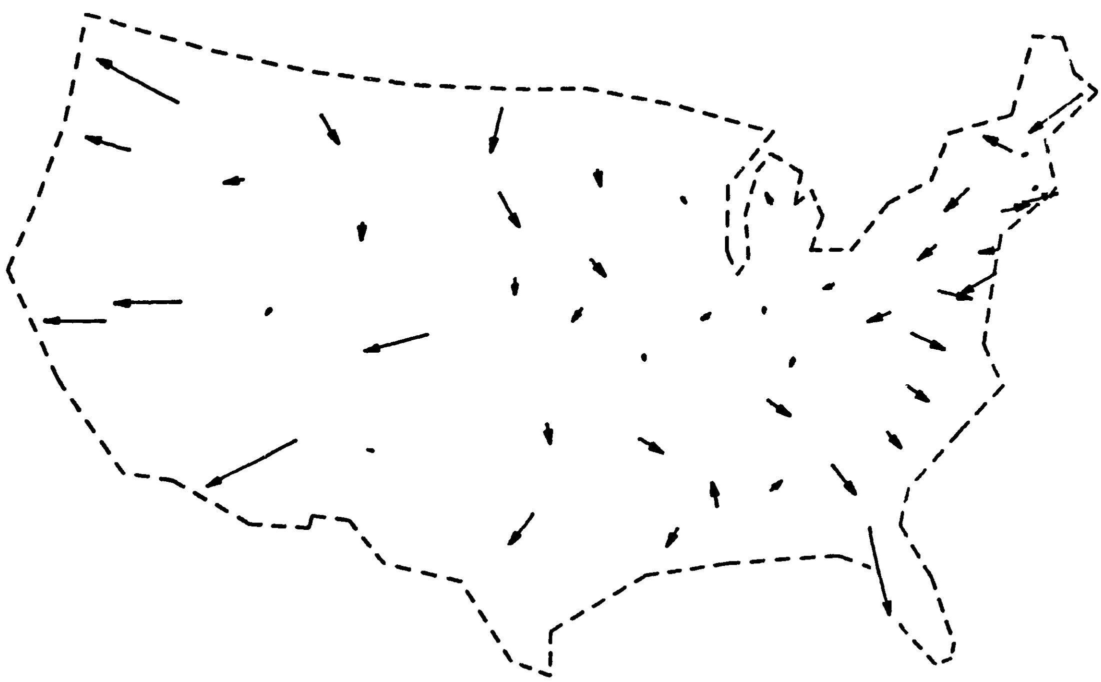

Figure 8: Winds of Change: U.S. Interstate Migration, 1965-1970

location to every other point, somewhat like a diagram of winds. If one takes a particular resultant of these arrows, one obtains a field of vectors which in fact quite adequately portrays aspects of the original data; see Figure 8. Such a vector field can be further analyzed mathematically to obtain the corresponding potential field; Figure 9. The technique is described in greater detail elsewhere.[15] As before it can only be justified if it is based on a clear mathematical model which bears some resemblence to the process being examined. No advance in cartography can be achieved without such a model.

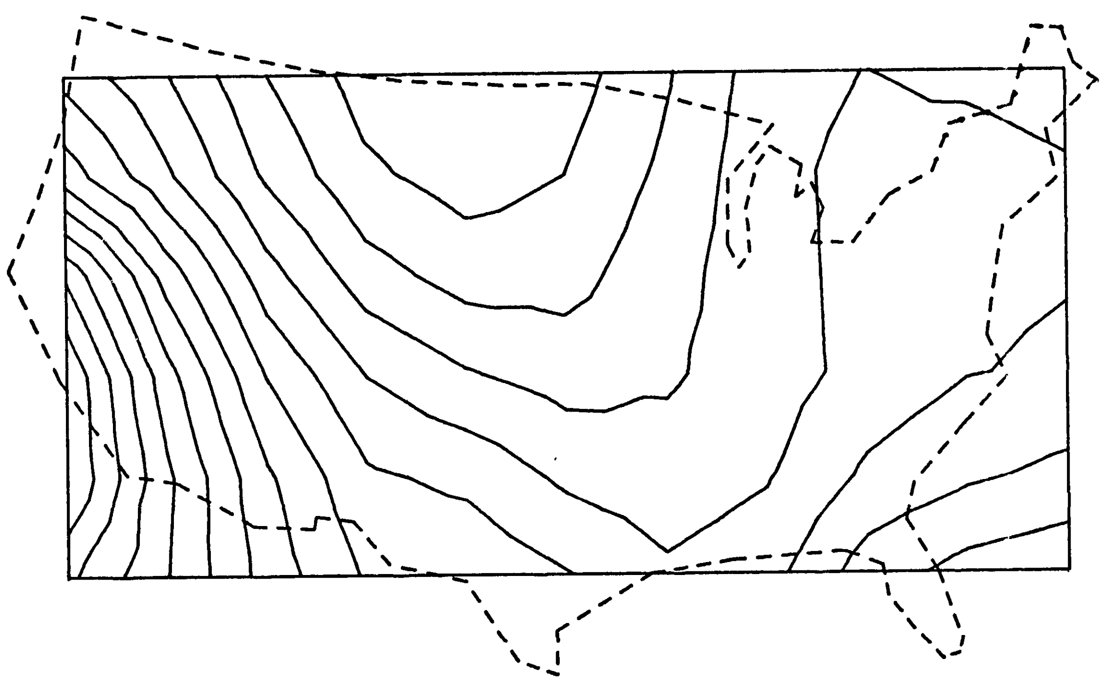

Figure 9: Interstate Migration, 1965-1970; the Forcing Function

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