TOPOLOGICAL PRINCIPLES IN CARTOGRAPHY

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ABSTRACT CHARACTER OF A MAP

A map may be described as a linear graph embedded in a orientable, two-dimensional manifold. A manifold, aside from being a two-dimensional continuum, is locally flat, that is, each point of the manifold is contained in a neighborhood homeomorphic to a disk.

A linear graph so embedded will, if it contains 1-circuits, delineate a number of simply connected domains. The collection of 0-cells and 1-cells of the graph, together with the 2-cells so delineated forms a two-dimensional complex, having the special property of local flatness.

The structure of this two-dimensional complex is completely specified by a pair of relations among the cells. These are the oriented incidence relations between 0-cells and 1-cells, and those between 1-cells and 2-cells. The local flatness condition implies that there is a form of symmetry, known as duality, between these two relations. A widely used model of this relational system, the DIME encoding, illustrates this symmetry in a perspicuous manner. This representation consists of a quadruple, or pair of ordered pairs of cell identifiers,

a, b; c, d

one such quadruple for each segment of the graph generating the map. The geometric interpretation of this model is as follows: a and b are interpreted as identifiers of the ordered pair of 0-cells, the initial and terminal points of the directed segment; c and d are interpreted as identifiers of an ordered pair of 2-cells, the right and left 2-cells separated by the segment. This latter interpretation is permissible only because of the local flatness condition of the manifold.

The natural dualism is apparent from the DIME representation. Exchanging the first and second pair provides a representation of a dual segment which can be drawn in a canonical way on the same manifold. The dual elements and relation pairs are indicated below:

0-cells	2-cells
1-cells	1-cells
boundaries	coboundaries*

The order relations on the oriented manifold are also expressed by the DIME representation. If we exchange the elements of the first pair, we obtain a representation of a segment oppositely directed on the oppositely oriented manifold, and

*The converse of the relation bounds is known as cobounds.

if we exchange the elements of both pair, we obtain a representation of the oppositely directed segment on the manifold with orientation unchanged. Since this is to all intents and purposes the identical segment as originally given, we see that the segment is invariant to an exchange of the elements within each pair. This fact allows us to establish a unique segment representation within a set of such quadruples. This uniqueness is established by ordering the first pair of identifiers in lex order, exchanging, if necessary, both elements of each pair. In case the elements of the first pair are identical, the ordering is established by the second pair. If both pair of identifiers are identical, the segment is not orientable.

Given a map abstractly defined in this manner, the description of the geometrical properties of the map are completed by a metrical representation of the manifold and the embedded graph. In practice, such a description is carried out to some specified degree of approximation. The fundamental encoding of the distinguished points of the manifold and the arcs establishes the degree of fidelity of the representation from the beginning.

It is customary to provide the basic data describing both the manifold and the embedded arcs as parametric arcs or data strings

$$x(s), y(s), z(s)$$

for a discrete parameter, $\underline{s}$. This description covers such structures as contours, profiles, street segments, and any other one-dimensional structures required for the representation of the relevant map data.

FINITE NEIGHBORHOOD SYSTEMS

Although the basic manifold has the usual complete set of neighborhoods of a two dimensional manifold, the finite representation represented by the map data must be regarded as a finite approximation to the complete structure. However, for a given base encoding, no finer description is available. The finest neighborhood system definable by the coded elements will be referred to as a set of fundamental neighborhoods.

The relational system represented by a DIME segment defines implicitly a set of four operators, which relate boundaries and coboundaries to individual cells. The designations of these operators are given in the table below.

E_0^1	from 1-cells to 0-cells
E_2^1	from 1-cells to 2-cells
E_1^0	from 0-cells to 1-cells
E_1^2	from 2-cells to 1-cells

The importance of these operators will be indicated by the following formulae for the retrieval of the fundamental neighborhoods.

The formula $E_2^1 E_1^0 \quad (\quad c_0 \quad)$

represents the operator composition which acting on a given 0-cell retrieves both the 1-dimensional and 2-dimensional elements abutting the cell.

The formula $E_2^1 \, E_1^0 \, E_0^1 \quad (\; c_1 \;)$

represents the operator composition which retrieves the cells abutting a given 1-cell.

The formula $E_2^1 \, E_1^0 \, E_0^1 \, E_1^2 \; (\; c_2 \;)$

represents the operator composition which retrieves the cells abutting a given 2-cell.

THE CONNECTION BETWEEN NEIGHBORHOODS AND GRAPHS

Since each fundamental neighborhood of a subgraph of the complex is a set of cells, we can represent such a neighborhood as a dual graph, and the boundary of such a neighborhood as a primal graph. We can therefore employ the theory of linear graphs in the analysis of neighborhoods.

For this purpose, ARITHMICON, is furnished with a graph analyzer consisting of a pair of routines, CHAINS, and KIRCHOFF. The first of these routines reduces a graph by eliminating vertices of index two. The resulting graph will be referred to as the reduced graph.

Segments within a graph can be classified as cyclic or acyclic. A segment is cyclic if it belongs to some proper 1-circuit of the graph. If a segment is acyclic, its removal will separate a connected component into two pieces.

Segments can also be retracted. This is not a classifying property of a segment, but of a graph. By retraction, a graph can be reduced to a set of loops.

The routine KIRCHOFF provides all the information necessary to classify segments, count loops, and to retract segments within loops where this is possible. The resulting graph is a Hsumi tree or a cactus.

It has not been found necessary to utilize all the information provided by KIRCHOFF. A curtailed graph code consisting of a quadruple of integers is returned by the system ARITHMICON graph analyzer,

$$n_1, n_2, n_3, n_4$$

The interpretation of these numbers is as follows

n_1 denotes the number of independent loops of the graph

n_2 denotes the number of segments of the reduced graph

n_3 represents the number of acyclic segments of the reduced graph

n_4 represents the number of vertices of the reduced graph

Some of the codes will be given for the fundamental neighborhoods and their more commonly encountered anomalies. For a fundamental open neighborhood of a

0-cell, the admissible graph codes are

n+1,n+1,0,1	for points on the boundary of an elementary 2-cell
n, n, 0, 1	for points interior to an elementary 2-cell

a code representing a graph consisting of a set of loops having a common point.

In cases in which a graph falls into pieces, KIRCHOFF returns a code for each individual component.

In the foregoing example, n represents the number of singular segments abutting the node, that is segments embedded in an abutting 2-cell.

Any other code indicates some anomaly due to a coding error. For example

0,1,1,2

is a common anomaly representing a graph consisting of a single 1-cell with two distinguished endpoints.

Another frequently occurring anomaly is

1,2,1,2

which represents a graph consisting of a loop with a tail, and two distinguished vertices.

The occurrence of components invariably indicates the inadvertent replication of a 0-cell identifier.

APPLICATIONS OF THE PRINCIPLE OF HOMOTOPY

The analysis of graphs outlined in the preceeding section relates to homeomorphic classes of graphs. These classes represent graphs that can be thought of as abstractly identifiable under a relabeling. However the requirements of homotopy relate to the continuous deformation of one figure into another within the confines of the manifold. The homotopy classes therefore distinguish between the interior and exterior areas of bounded domains.

There are two important applications of this principle. First, the fundamental neighborhood of a point must contain that point within its interior. If a point in polygon test is made between a given point and the cyclic boundary of its fundamental neighborhood, any homotopic inconsistency of the metric description can be detected. This is one of the most fruitful tests for uncovering errors in coordinates.

The second application is somewhat more complex. Consider a system of contours describing the shape of a two-dimensional manifold. Some observers have pointed out that such a system of contours can be ordered by set inclusion. A more general ordering principle is available. If we consider the set of contours to be generated by the intersections of the manifold with a one-parameter family of parallel surfaces, we can regard the contours in the neighborhood of a given contour as homotopic images.

Thus a contour lies between two of its neighbors if it is an intermediate contour encountered as the parameter varies between two of its values. There is therefore a mapping between open intervals of the parameter and the contours themselves. The end points of these intervals correspond to the singular curves of the system of contours. This observation implies that the system of contours can be mapped onto a linear graph. Such a mapping provides a useful organizing principle for the storage and retrieval of contour data.

There is some tendency to prefer storage and retrieval of such descriptive data from files representing systems of profiles. The identical mapping described above pertains to a data file of this kind. The mapping is characteristic of the abstract nature of the data, and not of the particular form of representation.

DIMENSIONALITY IN COMPUTERIZED CARTOGRAPHY

This presentation will close with a brief mention of the subject of dimensionality. To begin with, practically all feasible applications of cartography as it exists today are concerned with the description of two-dimensional manifolds. Although these manifolds are imbedded in three or higher dimensional spaces, they remain two-dimensional objects.

In order to find common examples of practical applications of the geometry of three-space one would have to leave the domain proper of cartography and enter that of engineering constructions.

When the need to consider legitimate three-dimensional structures becomes established, it will be found that the theory just outlined, which pretains to the two-dimensional manifold, will fit exactly without change into the more inclusive theory. The more inclusive theory will differ only in the introduction of a broader relational system, essentially the system obtained when the local flatness condition is dispensed with. However, the theory will apply unchanged to those subsystems which are representable as two-dimensional manifolds within the more comprehensive system. It would be too great an excursion to enter into greater detail about this matter here. However it is important to realize that the system just described will not have to be discarded in order to accommodate the needs of the coming era in which higher dimensional representations will be required.