A SURVEY OF THE MATHEMATICS OF MAPS

Marvin White
Application Mathematics Research Staff
U.S. Bureau of the Census

Overview

The automation of cartography demands the explicit analysis of maps and elaboration of their structure. Previously, cartographers dealt intuitively with these matters and developed their intuition through study and apprenticeship. Today, the application of mathematical theories is required so that the goals of automation may be achieved without sacrificing the informative character that cartographers intuitively infused into maps. This is not to say that we can automate cartographic intuition, but that we can avoid frustrating it, if we are careful.

A study of the nature of maps leads one to a variety of mathematical fields. Topology addresses the strongly held intuitive ideas we have regarding the nature of space and dimension.

In graph theory one studies linear networks, which occur in maps in many ways. Boundaries of regions, transport routes and tributary systems may all be regarded as graphs. A point of connection between graph theory and topology, namely the study of planar graphs, is particularly interesting. Here we see that both fields share a symmetry called "duality," having important practical implications.

More abstract structural features of maps are in the domain of lattice theory and hypergraph theory. Lattice theory involves partial orderings, such as the nesting and overlap of regions, and hypergraph theory is a generalization of graph theory that has been applied to the problem of representing maps.

The metrical features of maps, distances, angles, areas, etc., are studied in analytic and projective geometry. These fields are traditionally learned by cartographers as a basis for projections and scaling. No more will be said regarding these topics, since they are already well known.

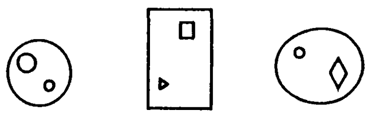
An applied mathematician endeavors to simplify and clarify a theory and then transform theorems into algorithms. The importance of abstract algebraic theories cannot be over-emphasized for simplication and clarification. Without them, one's ontology may inflate to ponderous size and computer programs are correspondingly patched and disorganized. In the following pages, we examine portions of these theories relevant to cartography and emphasize the connections among the various mathematical fields.

Topology - the Foundation for Automated Cartography

We look first at topology, since it provides a foundation for automated cartography, just as our intuition is the foundation for manual cartography. This discussion occupies most of the paper. The three figures in figure 1 are equivalent in the sense that each may be continuously modified into the others. Topology is concerned with the properties that the figures share, i.e., with what remains unchanged under continuous deformation.

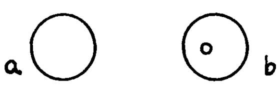
Homeomorphism

A topological transformation, also called a homeomorphism, is a continuous deformation, intuitively a rubber sheet transformation, where neither rips nor folds are permitted. A precise definition of homeomorphism in terms of mappings and continuity can be found in any text on topology but the intuitive idea will suffice here. The three figures in figure 1 are homeomorphic but the two figures in figure 2 are not, because a cut must be made in the interior of a to deform it into b. Alternatively, the points on the boundary of the hole in b must coalesce into a single point (a singularity) to transform b into a. This would also violate the requirements for homeomorphy.



TOPOLOGICALLY EQUIVALENT

Figure 1.



TOPOLOGICALLY DIFFERENT

Figure 2.

Cells and Complexes

Armed with homeomorphism, one can deform simple objects and combine them to create complex structures. This is the approach of combinatorial topology. The building blocks of combinatorial topology are called cells. A 0-cell is a point; a 1-cell is a simple curve; a 2-cell is a disk or rubber sheet deformation of a disk. In general an n-cell is a homeomorph of an open n-dimensional spheroid, i.e.

$$\{x \in R^n \mid d(x,0) < 1\}$$

The boundary of an n-cell is an (n-1) - circuit. The boundary of a 2-cell (a disk) is a 1-circuit (a circle) and the boundary of a 1-cell (an arc) is a 0-circuit (just a pair of points). An n-circuit is the homeomorph of an n-dimensional sphere, i.e.

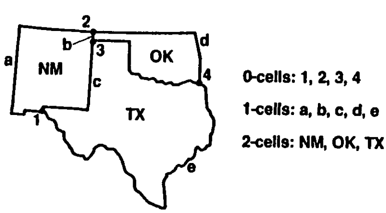
$$\{x \in R^{n+1} \mid d(x,0) = 1\}$$

N-cells, n-circuits and the boundary relation between them are all topological. Any n-cell is homeomorphic to any other n-cell. Any deformation of a disk produces a 2-cell (by definition), and the circle bounding the disk becomes the 1-circuit bounding the 2-cell, so the boundary relation is also topological.

We can now begin to describe a map in topological terms, i.e. in a way that does not depend on projections or scales or shapes. A map is a 2-dimensional complex, which is a collection of 0-cells, 1-cells and 2-cells such that:

1) Every 2-cell is bounded by 1-cells in the collection;
2) Every 1-cell is bounded by 0-cells in the collection;
3) Every 0-cell is on the boundary of some 1-cell;
4) Every 1-cell is on the boundary of some 2-cell.

Figure 3 illustrates the topological description of a map. There are three 2-cells, six 1-cells and four 0-cells. Also, the 1-cells bounding a 2-cell form a 1-circuit.



A MAP IN TOPOLOGICAL TERMS

Figure 3.

By merely describing a map as a collection of topological objects, we have done more than provide an esoteric vocabulary. We have described features that are utterly independent of measurements, coordinates and shapes, which are structural features of the map and thus are important for representing the map in a computer.

Surfaces

Now we can synthesize a representation of a map by combining n-cells, but our topological description of a map is not yet complete. A map is not an arbitrary 2-dimensional complex, rather it is a smooth 2-dimensional surface. In fact maps are almost always drawn on planes or spheres, imposing further topological restrictions.

On a smooth surface (or manifold) every point is contained in an open disk and this restricts the way in which n-cells may be combined. It can be proven that this restriction is equivalent in combinatorial terms to:

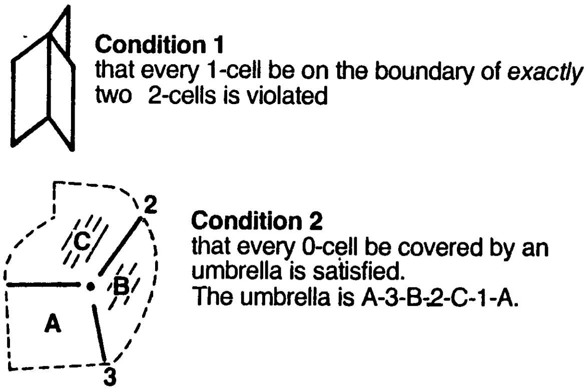
1) Each 1-cell is incident with exactly two 2-cells.
2) At each 0-cell there is a unique umbrella, i.e. a cyclic alternating chain of 1-cells and 2-cells.

The two conditions are stated imprecisely here but figure 4 will make the idea clearer. For a full treatment the reader is referred to standard texts on combinatorial topology. Anyone familiar with Dual Independent Map Encoding (DIME) will recognize that condition 2 is tested in the DIME node edit, and that condition 1 is automatically satisfied.

Dimension

Very few cartographic maps represent 3-dimensional objects. The surface of the earth is 2-dimensional; to determine a point only 2 coordinates, longitude and latitude, are required. Although altitude may also be specified it merely describes the embedding of the 2-dimensional object in a 3-dimensional space.

3-D is much more complicated than 2-D. For example, knots are possible in 3-D but not 2-D.



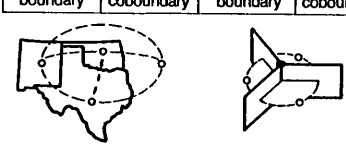
NECESSARY CONDITIONS FOR A SURFACE

Figure 4.

Duality

Although we are still discussing topology, it is well to note that duality is a symmetrical relation that occurs in many parts of mathematics. We are concerned with duality in topology (Poincare duality) and in graph theory (geometric and combinatorial duality). The Poincare duality is shown for 2 and 3 dimensions:

2-D		3-D	
Primal	**Dual**	**Primal**	**Dual**
0-cell	2-cell	0-cell	3-cell
1-cell	1-cell	1-cell	2-cell
boundary	coboundary	boundary	coboundary

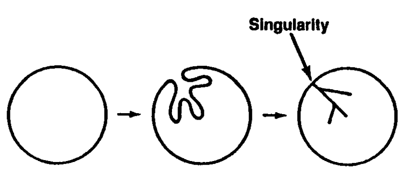


Poincaré Duality

The principal of duality is that if in a theorem about n-cells or boundaries and coboundaries, one replaces every occurrance of a primal by its dual and visa versa then the new statement is also a theorem. In automation it means that any topological structure or facility provided for a primal must also be provided for a dual and visa versa. This is actually a great advantage, since it halves the number of programs we need for a large class of topological calculations, such as programs for chaining and computing incidence relations. It is this symmetry that spawned the DIME code, which itself is symmetrical in the same way.

Extensions

It is useful on occasion to extend a theory to accomodate an application, rather than force the application to fit existing theory. Corbett extended topological theory outlined above to include singularities, to model cartographic features interior to 2-cells. Figure 5 shows how an embedded tree is constructed by a singular transformation, rather than a homeomorphism.



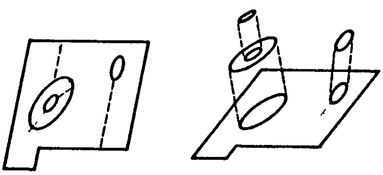
FORMING A TREE BY SINGULAR TRANSFORMATION

Figure 5.

The singularity occurs when distinct points coalesce to become one, violating the smooth deformation requirement for a homeomorphism.

Corbett also generalized 2-cells in the model to 2-cells with holes, as though parts had been removed with a cookie cutter. This was to accomodate cartographic features like lakes and islands in lakes without introducing artificial cuts from the border to the hole. The cuts are needed to make the boundary of a region a single circuit. One would traverse the cut once from the outer boundary to the hole and again in the reverse direction.

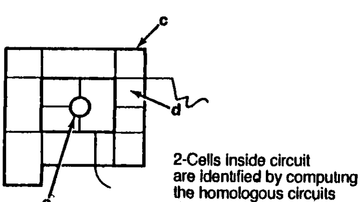
The extension to 2-cells with holes was accomplished by including homology theory. The homology group classifies disks with holes by counting the holes. Referring to figure 6, we see that even complicated arrangements of lakes and islands are treated directly in homology theory without recourse to artificial cuts.



HOMOLOGY THEORY AVOIDS ARTIFICIAL CUTS

Figure 6.

An example of a useful calculation involving homology is determining what regions are interior to an arbitrary 1-circuit topologically - not by point in polygon algorithms. Figure 7 illustrates this computation.



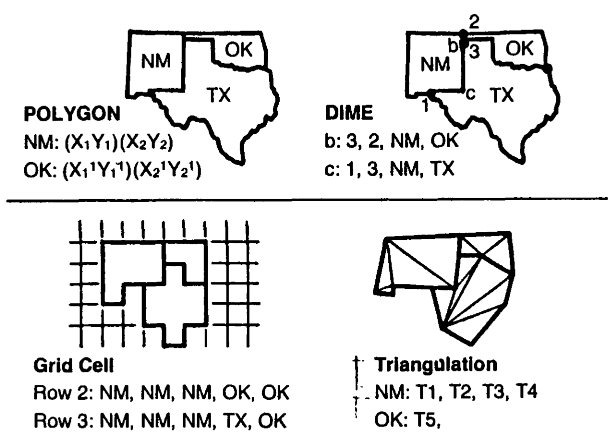
HOMOLOGOUS CIRCUITS

Figure 7.

One merely makes a counter clockwise walk around c collecting the 2-cells on the left. The boundary of the collection is both c and the homologous circuit d. To continue d must be traversed to collect 2-cells interior to it. Ultimately the entire 2-cell is filled in. This completes the outline of the topological theory. We next look at map encoding techniques and then proceed to graph theory, etc.

Topological Character of Map Encodings

It is instructive to examine current map encoding methods in light of the topological theory of maps. Figure 8 illustrates the methods. The most primitive encoding is a tracing of the lines and possibly annotation of a map. Generally, the traced lines will not connect resulting merely in a set of 1-cells with associated metrical data.



TOPOLOGICAL CHARACTER OF MAP ENCODINGS

Figure 8.

If one goes to the trouble of connecting the lines, e.g. by forcing end point coordinates to agree, then the topological structure is a linear graph, which is a great deal more.

A similar method is to trace region boundaries forming a 1-circuit for each region. Again, common boundary lines will fail to match, unless one goes to the trouble, a great deal of trouble in fact, to force them to match.

In the former case one has a collection of 2-cells with a metrical description of their boundaries, in the latter much more topological information. To the extent that boundaries match, one can identify 1-cells and 0-cells.

The DIME method is based on the topological theory, taking particular advantage of duality. In DIME each 1-cell is coded along with its two bounding 0-cells and two cobounding 2-cells. Of course, each n-cell may have its metrical description coded also--for a 0-cell its coordinates, for a 1-cell or a 2-cell its shape, subject to the metrical description of its boundary.

A grid cell description of a map forces region boundaries to conform to grid cell boundaries, so we have a set of highly regular 2-cells with implicit adjacency relations and no particular 1- and 0-cell data. Sometimes the existence of essential 1- and 0-cells can be inferred from the grid data, however, it is impossible in a grid cell map to represent the intersection of five or more lines. The advantage of gridding is that there are available many picture processing algorithms and some algorithms, e.g., merging two maps, are greatly simplified.

Finally, triangulation is used by many digital terrain models also because of the existence of certain useful algorithms such as contour interpolation. Here we have a simplicial complex, i.e., a complex of simplexes (triangles) rather than the more general cellular complex of 0-, 1- and 2-cells. Triangles are highly regular 2-cells. Indeed, the topological theory of complexes is usually developed starting with simplexes and generalizing to cells.

One pays for the algorithmic simplicity of highly regular cells in an inability to conservatively extend the theory. Fusing grid cells or triangles does not generally produce grid cells and triangles but fusing 2-cells does yield 2-cells. So one can not apply the theory or the programs to constructed objects, like neighborhoods, in the case of regular cells.

Graph Theory

Graph theory, like topology, is a rich source of mathematical ideas useful in cartography. The usefulness is enhanced by the more general applicability of graph theory to computation and operations research, which has motivated much investigation and produced many algorithms, such as minimum path algorithms.

Obvious applications of graph theory to cartography and related topics are the flow problems, like computing hydrologic and transport flows. Many not so obvious applications are related to planarity and duality. This connection to topology sheds light on both subjects.

A graph is a set of points and lines such that each line is terminated by points in the set. In a directed graph the lines also have direction. The 1-skeleton of the 2-dimensional complex is a graph.

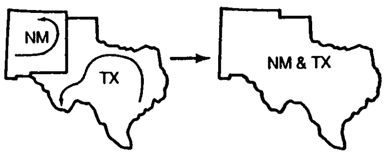
A planar graph can be drawn on a plane or sphere with no line intersections. For cartography the most useful characterization of planarity is MacLane's, who solved the problem by proving that a graph G is planar if and only if it contains a certain number of loops L(1), L(2), ..., L(n) and

A. Every arc of G belongs to exactly two loops;
B. The only independent relation satisfied by the loops is

$$\sum L(i) = 0.$$

Figure 9 illustrates MacLane's theorem and the adding of loops.

Now the connection to topology becomes clearer. Here condition A is the same as condition 1 that a complex be a surface. The pair of conditions A and B imply conditions 1 and 2. So, the MacLane's test for planarity is very much like the topological test for smoothness, but planarity is a bit stronger.



Addition of loops
in MacLane's Theorem.

GRAPH THEORETICAL ALGEBRA OF LOOPS

Figure 9.

The connections between graph theory and topology are strong. The geometric dual of a graph is just the 2-D Poincare dual of topology. The requirements for a smooth surface correspond closely to the planarity requirements. Many of the topological tests, like the DIME node edit (umbrella test), are just graph theoretical analyses of the 1-skeleton or the dual 1-skeleton of the 2-dimensional model.

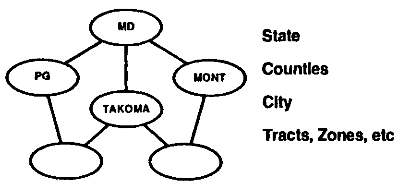
Indeed, a very large part of the topological theory can be reformulated in graph theoretical terms, and this, largely by substituting vocabulary. We would say, for example, that a DIME file is an encoding of the graph and its dual, rather than 1-cells and their incidence relations.

Lattice Theory

We now turn to lattice theory, which is even more abstract than graph theory. In the United States, the nation contains the states and the states contain the counties and the counties contain the cities sometimes, but sometimes cities span county boundaries. The "sometimes" destroys the hierarchy, resulting in an unwieldy partial order. The order is the order of containment and it is partial because some pairs of elements are not related by set containment.

Partially ordered sets with certain restrictions are the subject of lattice theory. There are many examples of posets in cartography such as the nesting and overlap of regions and networks. A Hasse diagram, as in figure 10, represents a poset in a way that suggest an economical storage scheme for data files.

Such a lattice provides a useful control structure for geographical and related operations, such as tabulating data by a variety of geographic identifiers.



A HASSE DIAGRAM

Figure 10.

It is also worth noting that in lattice theory one can state a large part of the topological model but not as much as in graph theory. If the n-cells are regarded as closed point sets, i.e. sets containing their boundary, then the lattice theoretical connections among the cells are the boundary and coboundary relations, but without orientation.

Completeness

The reason for developing and applying these theories is to provide an information system to support automated cartography. The test of an information system is whether it answers the questions we need answered.

An information system modelled on the theories discussed above will be able to answer questions about the nature of the modelled map, (is it a sphere?), about incidence and adjacency relations, (does Lichtenstein share a border with Switzerland?), about the valency of nodes (how many streets intersect at a point?), about distances, sizes and angles (what is the area of Alaska?), etc. Because of the breadth of these theories and our care in applying them, we can have confidence in our ability to answer any question that could be answered by examining an ordinary map or set of maps.

To answer the mathematical question of completeness, i.e., can all possible questions regarding the map be answered, would require formalizing the theory in the manner of logicians. Even then we would be faced with the incompleteness inherent in arithmetic demonstrated by Goedel. So we proceed less formally, in the manner of most mathematicians, and satisfy ourselves that we have captured all the known phenomena.

Annotated Bibliography

The references are categorized and ordered alphabetically by author within category. The remarks are merely my interpretations. No attempt at completeness has been made, however, each category is covered comprehensively to the best of my knowledge.

General

Newmann, James - The World of Mathematics
This is a collection of expository papers in which many discuss geometrical ideas.

Mandelbrot - Fractals
Discusses space filling curves, brownian motion, and many other topics in terms of Hausdorf dimension.

Topology

Alexandroff, Paul - Elementary Concepts in Topology
This little book was written as a companion to Hilbert and Cohen-Vossen and gives a clear concise statement of the important ideas in combinatorial topology - simplexes, cells, dimension, manifolds, homology, etc.

Corbett, James - Topological Principles in Cartography
The topological theory of maps is presented in detail along with a computer language for implementing the theory.

Hilbert and Cohen-Vossen - Geometry and the Intuition
Lefschetz, Solomon - Introduction to Topology
Leftschetz, Solomon - Applications of Algebraic Topology
Develops relevant topics in topology, graph theory and duality and applies them to circuit theory, surfaces and planarity.
Moise, Edward - Geometric Topology in 2 and 3 Dimensions
A text intended for graduate students in mathematics. Moise takes care to alert the student to pitfalls and difficulties.
Veblen, Oswald - Analysis Situs
An early treatise in topology. Veblen develops the incidence matrix method of topological computation, much like methods used for DIME.

Graph Theory

Briggs, Lloyd and Wilson - Graph Theory 1736 - 1936
Original papers in graph theory and related topology with interpretations and historical remarks by B,L, & W.
Deo, Narsingh - Graph Theory with Applications to Engineering and Computer Science
Provides many useful algorithms and insight into graph theory from an applications viewpoint.
Harary, Frank - Graph Theory
A standard text.
Ore, Oystein - Graphs and Their Uses
An introduction to the elementary ideas with many examples.
Saaty and Kainen - The Four Color Problem
Graph theory and topology applied to the four color problem with an extensive bibliography

Hypergraphs

Berge, Claude - Graphs and Hypergraphs

Lattice Theory

Birkhoff, Garrett - Lattice Theory
Rutherford, D. E. - Introduction to Lattice Theory
A short text that introduces many lattice theoretical ideas.