COORDINATE FREE CARTOGRAPHY

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Introduction

There are three famous characterizations of planar graphs [1]. Kuratowski's identifies forbidden subgraphs K(5) and K(3,3). Whitney's demands a combinatorial symmetry called duality, and MacLane's requires the existence of a vector space of circuits with a particular algebra.

Although none of the three provides a construction of a planar representation of a graph, MacLane's is the foundation for Tutte's algorithm [3,4] and some later algorithms [1] for embedding a graph in the plane. These algorithms are used both as a test for planarity and as a means for drawing a graph that is planar.

Lefschetz [2] has proven MacLane's Theorem much more simply via recourse to algebraic topology. The algebra required by the theorem is quite easily understood topologically.

We proceed as Lefschetz did to greatly simplify Tutte's algorithm. This new algorithm is applied to a topological encoding of a map known as the Dual Independent Map Encoding (DIME) invented by Corbett [5].

The DIME scheme for maps is in widespread use throughout the world. In this scheme, incidence relations are explicitly coded and coordinates are associated with points and lines. The encoding of a line written as [from node, to node, left block, right block] may be tested for consistency.

The consistency requirements are severe enough that they can be used to guess coordinate values where they are in error or even where they have never been measured. The guessed coordinate values are useful for interactively presenting a display of a small portion of a map with erroneous coordinates. The clutter is eliminated and the console operator may proceed directly to the business at hand, e.g., locating street addresses or correcting errors.

Our Algorithm Applied

Our algorithm for drawing a map given only a DIME description without any coordinates produces a representation that is topologically consistent with the actual map (Figure 1). The guessed coordinates will be a rotation and/or translation of the original map, but they will allow for a consistent and uncluttered display that can be altered with a small amount of effort. Points can be relocated so that their position is closer to that of the original map. Using the coordinates of the relocated points and computing coordinates for all other points results in a likeness of the original map (Figure 2).

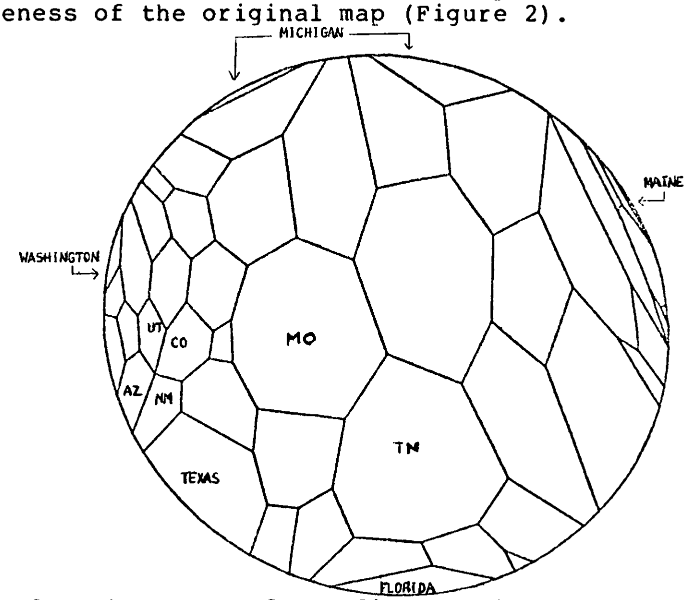


Figure 1. The computed coordinates yield a map topologically consistent with the original.

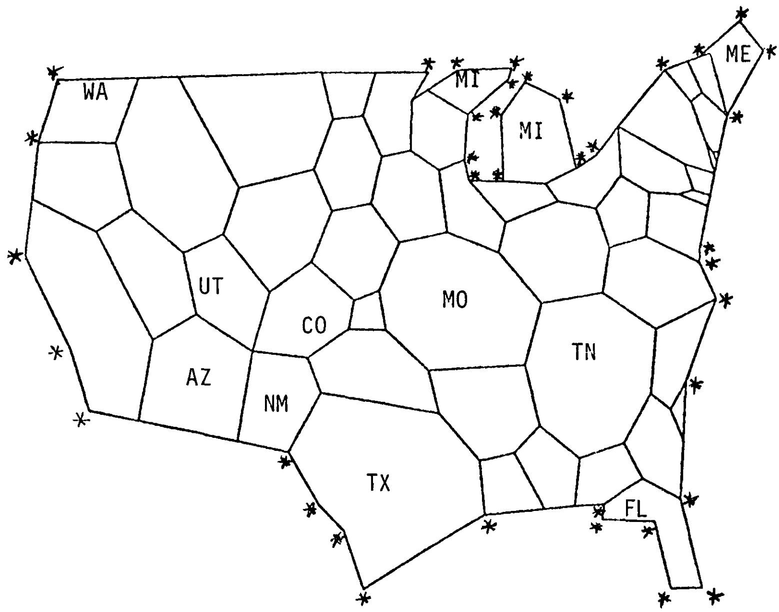


Figure 2. Repositioning boundary points (indicated by "*") produces a map similar to the original.

<u>Mathematical Character of a Map</u>

A map may be regarded as an assembly of elements of dimension 0, 1 and 2. This is a combinatorial view of maps. The elements are points, called 0-cells, line segments, called 1-cells, and areas, called 2-cells (Figure 3).

A 0-cell is a point	A 1-cell is a line stretched and formed, but not crossing itself	A 2-cell is a disk stretched and formed, but not torn or folded

Figure 3. 0-, 1-, and 2-cells

<u>The Algebra of Maps</u>

An algebra of maps is obtained from the elements of a map, the 0-, 1-, and 2-cells with coefficients from the field of integers. The interrelations among the dimensions is expressed in the boundary and coboundary operators.

A chain of 1-cells is written formally as

$$\sum c(i)\ b(i),$$

where c(i) is a coefficient and b(i) is a 1-cell. The boundary of 2-cell A in Figure 4 is:

$$\partial A = a + b + c$$

The boundary of B is:

$$\partial B = d + f + g - c,$$

the -1 coefficient indicates negative orientation, i.e., opposite to the the direction of the arrows.

Chains of 2-cells can be similarly expressed. The sum of A and B is just A + B. Notice that the boundary $\partial(A+B) = \partial A + \partial B$ is just the boundary of $A \cup B$.

$$\begin{aligned} \partial A + \partial B &= a + b + c + d + f + g - c \\ &= a + b + d + f + g \\ &= \partial (A \cup B) \end{aligned}$$

Note that the interior 1-cell c has coefficient zero in the sum. This algebra is the foundation of MacLane's Theorem and Tutte's algorithm interpreted topologically.

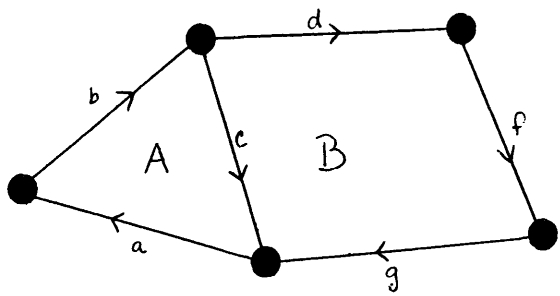


Figure 4. The algebra of maps

<u>MacLane's Theorem</u>

Theorem of Sanders MacLane [2, p.91]: Let G be connected and inseparable with Betti number R. A necessary and sufficient condition in order that G may be represented as a spherical graph is that it possess a set of R+1 loops L(1), L(2), ..., L(R+1) such that

I. Every branch of G belongs to exactly two loops L(h).

II. With a suitable orientation of the loops L(h), the only independent relation which they satisfy is $\sum' L(h) = 0$.

The statement of this theorem is combinatorial and without reference to topology. A spherical graph is also planar and visa versa, since the sphere may be projected into the plane stereographically [2, p.90].

Condition II implies that any R of the L(h) form a basis for a vector space of cycles in G. Condition I implies that the coefficients may be taken from the field of integers mod 2, ignoring orientation. The sum of two loops, L(h) and L(j), is then a single loop with the common branch of L(h) and L(j) omitted, or it is just the two loops again if they do not intersect.

Despite the lack of topology in the statement of the theorem, Lefschetz proves it topologically. He shows that each of the loops L(h) is the boundary of some 2-cell. Property I implies that the complex thus formed is a 2-dimensional surface, and property II implies that the surface must then be a sphere. Conversely, a spherical graph by definition has an embedding in the sphere. The Jordan Curve Theorem and results concerning the Betti numbers imply that there must be R+1 loops L(h) with properties I and II. For details the reader is referred to Lefschetz [2].

The important point about the topological proof is that it is very direct and appeals to one's geometrical intuition. If property I failed, as in Figure 5, we would not have a 2-dimensional surface. Rather we would have the intersection of two surfaces.

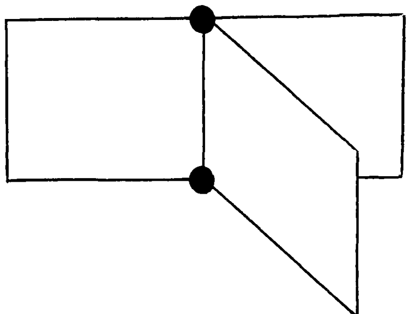


Figure 5. Intersecting surfaces fail to have property I

Tutte's Algorithm

This brings us to Tutte's algorithm. We quote his definition of "planar mesh," which is fundamental to the algorithm:

> A planar mesh of G is a set M=[S(1), S(2),..., S(k)] of elementary cycles of G not necessarily all distinct, which satisfy the following conditions:
>
> (i) If an edge of G belongs to one of the sets S(i) it belongs to just two of them.
>
> (ii) Each non-null cycle of G can be expressed as a mod 2 sum of some members of M.
>
> Condition (i) implies that no elementary cycle of G can appear more than twice as a member of M. Further, the mod 2 sum of all members of M is null.

Therefore, a planar mesh is just the graph specified in MacLane's Theorem but with orientation ignored.

The elementary cycles (also "peripheral polygons") of a planar mesh are the loops of MacLane's Theorem and boundaries of the 2-cells of Lefschetz's proof. In the span of two long and complicated papers, Tutte proves combinatorially that the planar mesh of a graph G may be identified, and presents his algorithm for drawing a graph using the planar mesh. Furthermore, he shows that for a nodally 3- connected graph there is a unique barycentric representation [3, p.759].

Tutte realizes the function mapping the graph into the plane as one that assigns cartesian coordinates to the nodes, and maps the edges to straight lines connecting the nodes [3, p.752]. The outer boundary is one of the elementary cycles; in fact, any one will do. The nodes of that cycle are assigned the coordinates of the vertices of a regular n-gon in the plane so that their cyclic order is preserved. The other nodes are interior to the n-gon and are assigned coordinates so that each node is at the center of mass of its adjacent nodes (see Figure 6). For a nodally 3-connected graph there is a unique assignment of coordinates satisfying those criteria. They are also determined by the following:

$$\sum_j{}' c(ij)\ x(j) = 0$$

$$\sum_j{}' c(ij)\ y(j) = 0 \text{ for } n < i \leq m \text{, where}$$

$$c(ij) = \begin{cases} -\text{ number of edges joining nodes } i \text{ and } j, \\ \quad \text{for } i \neq j \\ +\text{ valency of node } i \text{ for } i=j. \end{cases}$$

The coordinates for nodes $v(i)$, $1 \leq i \leq n$, are already known, as these are the vertices of the n-gon.

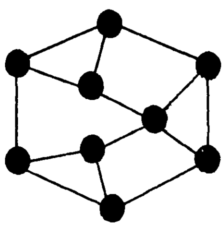


Figure 6. Barycentric representation of a graph

Our Algorithm

From the topological viewpoint, Tutte's algorithm is building a larger and larger disk rather than adding loops to form one large loop, which is the graph theoretical interpretation. A disk may be constructed vertex by vertex as well as block by block since the open neighborhood of a 0-cell is topologically equivalent (homeomorphic) to an open disk and a 2-cell is homeomorphic to a disk. The union of two overlapping disks is again a disk, as shown in Figure 7.

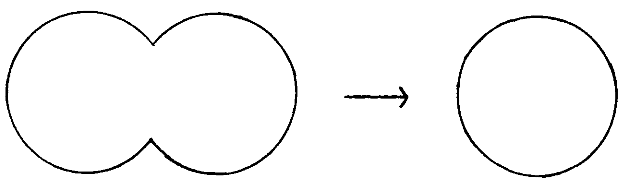


Figure 7. The union of two overlapping disks is equivalent to a disk

Disk construction continues merging overlapping disks until the entire graph is covered (Figure 8).

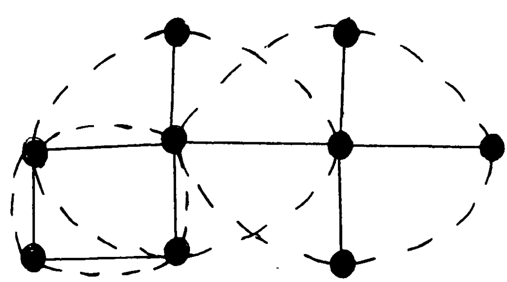


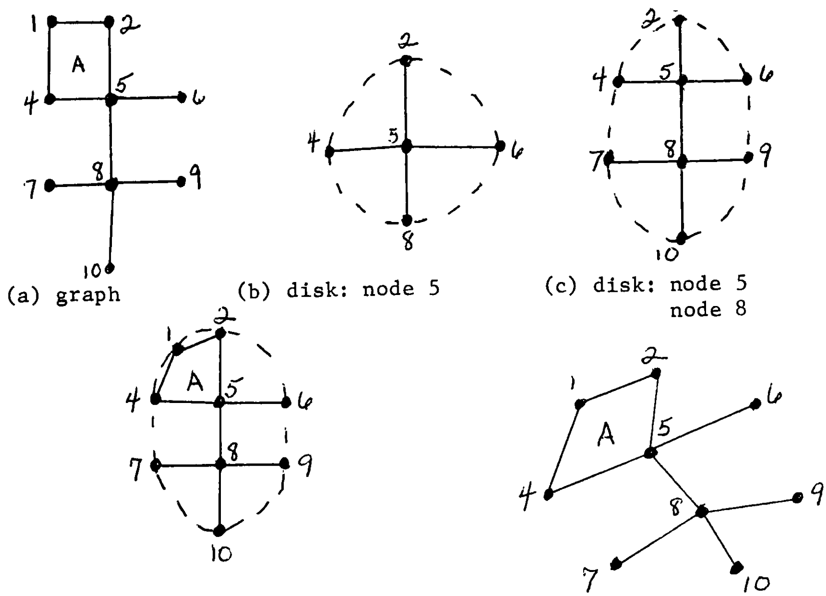
Figure 8. Disk construction

Tutte's algorithm and MacLane's Theorem both have restrictions on the connectivity of the graph. As our algorithm uses the topological equivalence between the open neighborhood of a vertex and a disk to allow the addition of nodes to the disk, the restrictions are not needed.

The two ways of building a disk, point by point and 2-cell by 2-cell, are combined in our algorithm. We proceed point by point as much as possible, since this simplifies identifying interior nodes. When an inconsistently coded vertex is encountered, it must be on the boundary, and incident 2-cells are then considered. The algorithm then continues point by point.

Given a segment list for a graph, the algorithm constructs disks and their boundaries, assigns coordinates for disk boundary nodes, and computes coordinates for disk interior nodes. Disk construction begins by identifying a consistently coded 0-cell as an interior node and its adjacent nodes as disk boundary nodes, or if there are not any consistent nodes in the graph, by identifying the boundary nodes of a consistently coded 2-cell as disk boundary nodes. The algorithm continues adding to the disk by examining disk boundary nodes in counter-clockwise traversal of the boundary. A consistent boundary node is replaced by its adjacent nodes, and added to the interior of the disk. If the boundary node is not coded consistently, then adjacent blocks are examined. Nodes on a consistent block boundary are added to the disk boundary, and the process continues by examining the next disk boundary node. The entire disk has been constructed when all disk boundary nodes have been examined, and the algorithm iterates to build any remaining disks for the graph.

Additions to the disk boundary occur in a manner which insures that the counter-clockwise traversal of the boundary is not disrupted. Thus, the disk grows in an orderly fashion, examining every disk boundary node exactly once. Node additions to the disk boundary are always inserted after the current disk boundary node and are ordered to maintain the counter-clockwise traversal. Additions to the disk boundary occur in two cases: the replacement of a consistent disk boundary node by its adjacent nodes, and the addition of boundary nodes of a consistent block. The node and block edit routines facilitate the maintenance of the counter-clockwise order of the disk boundary, as they chain the node co-boundary and the block boundary in counter-clockwise order. Figure 9 displays disk additions as the result of node and block edits, and shows the barycentric representation of the graph.



(a) graph (b) disk: node 5 (c) disk: node 5 node 8

(d) disk: node 5,8 block A (e) barycentric representation

Figure 9. Disk additions

Once a disk and its boundary have been constructed, coordinates are determined as in Tutte's algorithm. Coordinates of a regular n-gon are assigned to the disk boundary nodes, and are computed for disk interior nodes such that they are located at the center of mass of their adjacent nodes.

Bibliography

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