TOPOLOGICAL MODELS FOR ARCHITECTURAL AND ENGINEERING PROJECTS

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I. Introduction

The model presented here is intended for the interactive use of architects and engineers. The effective use of the model requires adequate computer support. In particular, it is desirable that adequate computer graphics facilities are in place so that selected structural elements may be displayed. A comprehensive system for projective geometry is also desirable. The purpose of the model is to provide facilities for the efficient evoking of selected images, and for the detection of structural or geometric anomalies. These anomalies will be of two kinds. First, there may be topological anomalies, whose presence makes consistent detailing impossible, and second there may be metrical anomalies arising from metrical inconsistencies with the topological structure. Such anomalies should be detected before fabrication is begun.

The model is a framework for organizing the project. It is constructed according to the principles of combinatorial topology. Here, a geometric object is considered as being a synthesis of elementary objects, cells, according to a specified scheme of interconnection. The cells and their connections are not to be regarded as representations of material objects, but rather as abstract geometric references to which the individual details are required to conform. The important property of this frame of reference is that it may be established at the outset as a self-consistent geometric object, before any detailing is begun. It thereby becomes a powerful tool for controlling the mutual consistency and self-consistency of the details of the structure.

II. Combinatorial Topology in Three Dimensions

Combinatorial topology deals with elementary geometric objects, cells, and a single binary relation on pairs of cells. This relation is known as incidence. In principle, a list of incident pairs of cells provides a complete and unambiguous description of a method of assembling cells into an integral structure. Cells are endowed with a property, dimension, specified by an integer, 0, 1, 2, or 3; and hence we speak of cells as 0-cells, 1-cells, 2-cells and 3-cells. We refer to cells by the symbols, c^k, in which the super-script denotes the dimension of the cell. The relation of incidence is expressed by the symbols,

(2:1) $$I(c^k, c^{k+1})$$

Note that the cell of lower dimension appears first in the pair. The first cell of the pair is said to bound the second, and the second cell is said to co-bound the first.

The term, cell, is quite a useful one. Although the intended interpretation is that 0-cells are points, 1-cells are arcs, 2-cells bounded simple two-dimensional manifolds, the term itself carries no connotation of a metrical nature, such as position, curvature, torsion, length, area or volume.

There is a second point of view from which the term is useful. Much of a designer's work is synthetic, connecting simple elements (cells), or modules (collections of connected cells), in particular ways to form integral structures. On the other hand a designer may begin with a single amorphous cell, and procede to create a structure by subdividing the space, using the separation properties of cells of lower dimension.

III. Relations of Order and Orientation

There are certain important properties of cells and their incidence relations that are most simply expressible in topological terms. The most important of these properties is orientation. Every cell, of whatever dimension, may be oriented in two ways, referred to as positive and negative. Orientation depends on order relations. For the 0-cell, which is intended to represent a point, one arbitrarily associates with each 0-cell a positive or negative algebraic sign.

For a 1-cell, we establish an orientation (direction) by ordering the pair of bounding 0-cells. The first 0-cell is taken as negative, the second, positive. The negative of this 1-cell is defined by reversing the order of the bounding 0-cells, and therefore the direction of the 1-cell. This relation with the bounding 0-cells is often referred to as the "from-to" relation.

Analogously, the 2-cell is oriented by establishing an order on its bounding 1-cells. The poitive 2-cell is taken to be that version for which the cell lies to the left of each bounding 1-cell. Obviously, this may require that some, or all, of the bounding 1-cells are negative. We will refer to this scheme of order and orientation as the <u>near side</u> view of the cell, and the opposite view as the <u>far side</u>.

Finally, the 3-cell is oriented in terms of oriented 2-cells on its boundary. The positive version of a 3-cell is that for which the cell lies on the far-side of each bounding 2-cell. Thus, it is possible to think of the positive version of a 3-cell as that seen from without, looking in, and conversely, the negative version as the same cell from the inside, looking out.

The method of orienting the 2-cell is in agreement with a convention for which a positive traverse of the boundary is taken in the anti-clockwise direction.

IV. <u>Duality, Boundary and Coboundary Operators</u>

The duality of a topological structure is a form of abstract symmetry. The concept is of extreme importance to the model. The scheme known as Poincare duality is used. Dual concepts are listed as pairs in the following table.

0-cell	3-cell
1-cell	2-cell
boundary	boundary

In the sequel many definitions will occur for which dimension numbers and the terms "boundary" and "co-boundary" appear. Any set of statements of this kind is related to a dual set of statements formed by replacing the dual elements from the table.

We now give definitions of the two most important algebraic operators associated with a topological structure the boundary and coboundary operators. It will turn out that these operators are dual to each other. The boundary operator will be denoted by the symbol, B, and the co-boundary operator, by the symbol G.

The boundary operator is defined in stages. The first step is to define the operator on the domain, c^k, the set of k-cells. The domain of the operator is c^k, and its co-domain the power set of c^{k-1}.

For any cell, c^k, $B(c^k)$ is the set of all (k-1) cells for which the relation

$$I\ (\ c^{k-1},\ c^k\ )$$

holds.

This operator is extended to the domain consisting of the power set of c^k simply by taking set theoretic unions.

In practice the operator is realized as follows: For each cell of the argument set, the set B(c) is constructed. The signs of the occurrences of individual cells are retained. The union of all such boundary sets is then reduced to an unduplicated list of cell names, in which positive and negative occurrences of the same cell are distinguished. Each cell is associated with an integer denoting the multiplicity of its occurrences in the amalgamated set of boundaries.

The coboundary operator is defined by a set of statements dual to the set defining the boundary operator above.

V. The View and the Section

We are now in a position to relate the formalism of combinatorial topology to engineering drawing practice. The view, illustrated in Figure 1, represents a 2-cell, and its complete boundary, that is, the set of bounding

1-cells, and the 0-cells on the boundary of these 1-cells.

The section, illustrated in Figure 2, represents a section through a 1-cell, its cobounding 2-cells, and the 3-cells cobounding this set of 2-cells.

The view and the section are therefore dual structures. We further annotate the view with the names of the 3-cells on the near and far sides of the cell, and similarly, annotate the section with the names of the "from" and "to" 0-cells on its boundary.

VI. Circuits and cocircuits

A set of k-cells for which the bounding (k-1)-cells each occur exactly twice and with opposite orientation is a k-circuit. The term cycle is often used in place of circuit, and in some branches of topology, the term, loop, is used.

The set of all k-circuits will be denoted by the symbol Z^k, and the set of all k-cocircuits, by Y^k.

A convention is required for 0-circuits, and the usual one defines a fictitious boundary, the sum of the coefficients of the 0-cells. Thus pairs of oppositely oriented 0-cells are 0-circuits, and pairs of oppositely oriented 3-cells are 3-cocircuits.

In some of the analytical procedures it will be required to identify the non-bounding circuits (and their dual structures). When this is required, the bounding circuits will be denoted by Z^k_B and the cobounding cocircuits by Y^z_G.

VII. The Algebraic Structure and its Model

We have now reached the final step in the description of the topological framework, the construction of the algebraic model. This model will provide for an efficient storage structure for the necessary data, and a natural and convenient language, L, for evoking images of particular designated structures.

The model is the algebraic structure, consisting of the elements,

(6:1) $$(C^k, Z^k, Y^k, B, G)$$

The symbols of the structure are incorporated into a logical language as an extension. In this extension one can express all of the necessary conditions to be satisfied by the model, and all of the necessary practical descriptions of cells, cell-modules, and extended structures that may be required.

VIII. Examples of terms in the language, L

The operator, $(1 - B)^{-1}$, will represent the Neuman series, $1 + B + B^2 + \ldots$.. When operating on cells this operator is always truncated, since c^{-1}, and c^4 are taken to be null sets.

To evoke the image, the view of a 2-cell, the term is

(7:1) $$(1 - B)^{-1} c^2$$

To evoke the section of a 1-cell,

(7:2) $$(1 - G)^{-1} c^1$$

The duality of these two structures, already remarked, is evident from this pair of formulae.

To annotate the 2-cell with its pair of cobounding 3-cells, the required algebraic term is

(7:3) $$G\, c^2$$

To annotate the 1-cell with its pair of cobounding 0-cells the term is

(7:4) $$B\, c^1$$

The conditions that must be satisfied for each cell of the model are,

(7:5) $$(c^k)\ B(c^k)\ e\ Z^{k-1}$$ (e denotes set membership)

and the dual condition

(7:6) $$(c^k)\ G(c^k)\ e\ Y^{k+1}$$

These examples illustrate the fact that all of the necessary terms and sentences can be expressed in standard mathematical terms within the language, L.

IX. One and two-dimensional structures

Many substructures in a complex project are most conveniently abstracted as one and two-dimensional geometric objects. Piping, electrical conductors, partitions, decks, etc. are obvious examples. Such structures are embedded in three-dimensional space, but are themselves abstracted to one and two-dimensional form.

The Poincare duality scheme in two dimensions differs from that for three dimensions. The dual pairs are listed in the following table.

0-cell ------ 2-cell
1-cell ------ 1-cell
boundary ---- boundary

For a one-dimensional object the table is

0-cell ------ 1-cell
boundary ---- boundary

The model for the one-dimensional case is trivial, but the model for the two-dimensional case is determined by the same algebraic structure as that constructed for three-dimensions. The difference is that the value of k, the dimension numbers are restricted to 0, 1, and 2. Thus although the structures are formally similar, the two dimensional model is simpler. These models are displayed as tableau in Figures 3 and 4.

X. Metrical descriptions

Each cell requires a metrical description. It is assumed that this set of algorithms, $A(c_i^k)$ is provided for each cell of the structure.

There are two general conditions that must be satisfied by these descriptions; 1) The non-intersection condition for pairs of cells, 2) The continuity condition between a cell, its boundaries and its coboundaries.

These conditions may be verified algorithmically, and even approximate interfaces discovered.

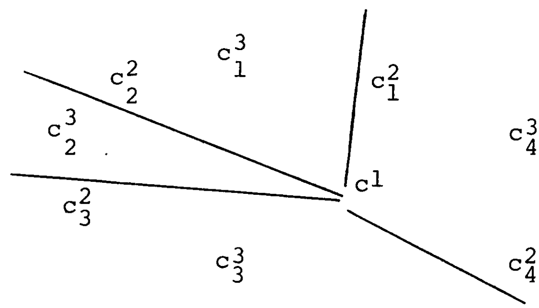
XI. The data storage structure

The data storage structure is shown schematically in Figures 3 and 4. Structures for both two-dimensional and three-dimensional manifolds are shown. It is evident that these storage structures are simple maps of the algebraic structure.

Seperate lists of the algorithms defining the shapes and sizes of the individual cells are provided. These are entered with the cell name as a key.

It is an absolute necessity that this structure be edited both topologically and metrically. The topological editing consists of verifying that the model satisfies the cyclic and cocyclic conditions for a boundary and a coboundary respectively. The metrical edit consists in verifying the continuity conditions for a cell, its boundary, and coboundary, and the non-intersection condition for all pairs of cells.

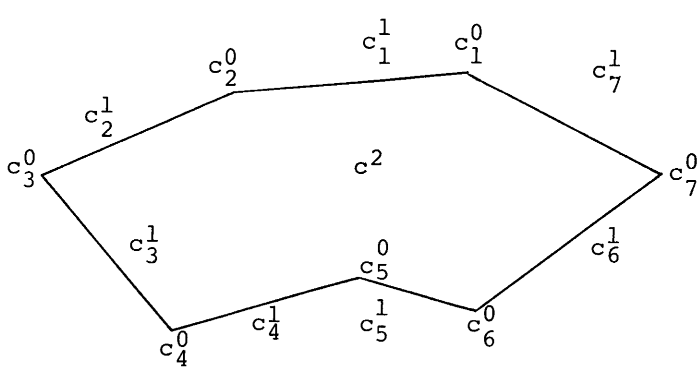
A data structure that passes these editing tests is necessarily self-consistent both metrically and topologically. The importance of conducting these tests interactively cannot be overestimated.
This form of correction permits re-editing of alterations immediately upon data entry. In fact, it is possible for a designer to create the entire structure under this discipline, so that no inconsistent data is ever accepted into the model at all.



looking from c_1^0 to c_2^0

<u>Section through a 1 - cell</u>

Figure 1



near side c_1^3
far side c_2^3

<u>View of a 2-cell</u>

Figure 2

C^0 $\quad$ Y^1 $\quad$ C^1 $\quad$ Z^1 $\quad$ C^2

$$(c^0 : y^1) \overset{G}{\text{-----}} y^1 \text{------} (c^1 : c^0_1, c^0_2, c^2_1, c^2_2) \text{-----} z^1 \overset{B}{\text{-------}} (c^2 : z^1)$$

Note that the elements, c^1, are self dual. The sets c^0, Y^1, are dual to the sets c^2, Z^1.

There are three entry points.

<u>File Layout for Model of a Two-Dimensional Manifold</u>

Figure 3.

$$C^0 \qquad\qquad Y^1 \qquad\qquad C^1 \qquad\qquad Y^2$$

$$(\, c^0 \colon y^1) \xrightarrow{\;G\;} y^1 \text{---------} (\, c^1 \colon c^0_1, c^0_2,\ y^2) \xrightarrow{\;G\;} y^2 \text{----------} C^2$$

$$C^3 \qquad\qquad Z^2 \qquad\qquad C^2 \qquad\qquad Z^1$$

$$(\, c^3 \colon z^2) \xrightarrow{\;B\;} z^2 \text{---------} (\, c^2 \colon c^3_1,\ c^3_2,\ z^1) \xrightarrow{\;B\;} z^1 \text{----------} C^1$$

Note that the data storage structure is an exact map of the data structure. The sets, C^0, Y^1, C^1, Y^2 are dual to the sets, C^3, Z^2, C^2, Z^1. A symbol followed by the symbol, ":", is an entrance key; there are four such keys in all.

File Layout for Model of a Three-Dimensional Manifold

Figure 4