TRIANGULATION-BASED TERRAIN MODELLING-WHERE ARE WE NOW?

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Introduction - Triangles versus Squares

In the last few years considerable effort has been expended by various individuals and organizations in the development of digital terrain modelling systems. Many different methods have been evolved, some of them better than others in technique, and some more suitable for particular data types and display requirements. Various authors have effectively surveyed the literature on the subject. Rhind (1975) divided the field into: i) zone partitioning (e.g., dividing the map area into regions whose vertices are defined by data points); ii) global fitting techniques, such as "trend surfaces"; iii) gridding methods where elevation values at nodes of an arbitrarily-defined grid are estimated by a wide variety of techniques, from rolling means to universal kriging; iv) "multi-quadric analysis" and v) contour chasing methods. Schut (1976) describes six groups: i) moving surface methods, requiring the computation of a surface at each data point, as for most weighted average techniques; ii) "summation of surfaces", in which he includes all methods involving the correlation theory of stationary random functions; iii) simultaneous patchwise polynomials, i.e. polynomials valid over "patches" of the surface, such that adjacent patches agree at their boundaries; iv) interpolation along equidistant parallel lines in a photogrammetric model; v) interpolation using a network of triangles with data points as vertices and vi) interpolation using characteristic terrain times.

It is not the intention of this paper to review further all of these techniques. Almost all of the above methods have in common an arbitrarily-defined regular grid into which are inserted the estimated elevation values prior to display by contouring or other methods. The exceptions are direct contour following, and triangulation techniques - the main subject of this paper. Consequently it is not inappropriate to make some general comparisons between "triangles" and "squares".

The general structure of the "squares" approach is clearly comprehended - which is why it has been so extensively used in a Cartesian society. The x-y values are implicit in the grid definition and only the elevations need to be stored. Indeed, using patchwise polynomials only the cofficients of each patch need be stored (e.g. Jancaitis, 1977).

Triangulation schemes, therefore, are methods that describe the relationships between the original data points. As such they are, and should be treated as, cartographic data structures. This is especially true when the data points are selected to represent features of the topography (e.g. Peucker, 1977; Males, 1977) rather than merely being randomly located. If triangles are indeed cartographic data structures then enough information should be preserved with each to permit negotiation of the network and the rapid answering of questions concerning the relationships between data points, (Peucker, 1977; Lawson, 1977). Gold et al. (1977) suggested preserving for each triangle, pointers to the three data points forming the vertices as well as to the three neighbouring triangles. This, or any equivalent scheme, permits the economical determination of any local question of relationships between neighbours by an economical <u>local</u> interrogation of the data structure. Computationally it is cheap, in storage costs it is expensive, requiring two triangles of six elements for each data point. Clearly this approach is not primarily intended for where the data point density is already too great for comfort, as in some remote sensing applications.

Techniques of Triangulation

In some applications (e.g. Males, 1977) the triangulation is defined manually and becomes part of the data. Usually, however, it is required that this operation be performed automatically. There are two major parts to this operation - firstly the requirements for a "good" triangulation of any point set must be defined, and secondly an efficient algorithm must be specified for the actual generation of the network.

Various criteria for a good triangulation have been defined. Most take four points forming the vertices of a quadrilateral and then decide which of the two possible internal diagonals is preferable. Criteria include maximizing the minimum height (Gold et al. 1977), maximizing the minimum angle (Lawson, 1977) and minimizing the diagonal length (Akima, 1975). Shamos (1975) mentioned that only the Delaunay triangulation (equivalent to Lawson's maximized minimum angle - see Sibson, 1978) has been shown to have a unique solution without testing every possible set of triangles. Within the last couple of years it has become accepted that this is the best criterion to use for triangle definition. This triangulation is the dual of the Theissen or Voronoi or Dirichlet tesselation in which any location on the plane is assigned to the polygon containing the nearest data point.

Having obtained a criterion for triangulation we need an algorithm to implement it. Shamos (1975) has shown that the Voronoi tesselation may be defined as O(n log n) time for n data points. Many published algorithms do not achieve this. Lewis and Robinson (1978) triangulate by splitting the plane; McLain (1976), Akima (1978) and Lawson (1977) triangulate by starting with a nucleus of points and attaching the best remaining point onto this growing network. Gold et al. (1977) start with a large enclosing triangle, locate a data point inside it, subdivide the triangle into three, find the triangle enclosing the next data point by a linear walk through the network, and repeat the process. Optimization is currently performed after each data point insertion, and is O(n) since an average of six switches of the diagonal of a quadrilateral are required for each point. Green and Sibson (1978) achieve a similar efficiency for generating a Dirichlet tesselation and claim high speed for it. Because they preserve the polygons, not the triangles, there is a potential 50% saving in pointer storage. However, the requirement for additional pointers to handle the unequal number of links to each polygon, together with the inability to preserve non-optimal tesselations / triangulations such as those formed by manual entry of surface-specific lines (Peucker, 1972) reduces this advantage.

One feature in this should be emphasized: the last two algorithms, together with a few others, produce fast triangulations. While speed comparisons between machines are difficult, a simple example should make the point clear. With the algorithm of Gold et al. (1977), the insertion of a data point into the triangulation, as well as its optimization, typically takes only 1.5 times as much computing time as the reading of the x, y, z values from punched cards or other medium. <u>For most practical purposes, the computing time for triangulation may be ignored.</u>

<u>Properties of Triangulations</u>

As already mentioned, the dual graph of the Delaunay triangulation is a Dirichlet tesselation of polygons with the properties of the "proximal map" of the SYMAP package. Thus the zone associated with each data point may readily be outlined by taking the perpendicular bisector of each triangle edge connected to the data point under consideration. This is useful for various purposes, including resource inventory based on drill hole information. The cost is clearly little more than that of generating the triangulation.

A very useful property of any triangle in any triangulation is that it may be the basis of a three-parameter homogeneous coordinate system (Gold et al. 1977). If linear interpolation is a sufficient approximation to a terrain surface then the elevation of any internal x-y location is obtained by weighting each of the elevation values of the data points at the triangle vertices with the appropriate area coordinate, and summing.

The ability to process map elements in some sequential order is of considerable value with current computer equipment. A triangulation may be treated as a binary tree structure with respect to any arbitrary direction ("north"). Any triangle must have either one or two south or downwards facing edges (ignoring the neutral case of a vertical edge). Thus, if triangles are to be processed from north to south the initial triangle must be followed by either one or two neighbours to the south. The second one, if present, is put on a stack for later retrieval and processing continues with the other neighbour. The end result is a triangulation "cut" along various edges to form a tree structure (Gold and Maydell, 1978). Triangles are processed in swathes and pen movement is greatly reduced for any drafting (e.g. contour lines) required in each triangle. The order is such that, looking from the north, frontmost triangles are always processed prior to those behind them. This is valuable in some forms of terrain display. In addition, at any stage in the map generation, the southern boundary of the completed portion of the map is monotonic in the east-west direction. This has proven to be valuable in the construction of contour lines that may take any path through the triangular mesh.

Interpolation Within Triangles

For purposes of contouring, it is usual to subdivide the triangle into a regular grid of subtriangles, (Gold et al., 1977), estimate elevations at each grid node and then trace the contours through the grid. This is similar to contouring a rectangular grid, but without the ambiguities due to interpolating between four corners.

As mentioned above, linear interpolation within each triangle is very rapid and economical. Where a "smooth" surface (continuous in the first derivative) is required, rather elaborate interpolation functions are needed. This is the most important field of current research, and is where most of the computing time is spent. The method used by Gold et al. (1977) is relatively economical but may be rather "stiff" for some applications. Akima (1975) uses a bivariate fifth degree polynomial, Lawson (1977) uses Clough & Tocher piecewise polynomials in each triangle. Powell and Sabin (1978) use a piecewise quadratic approach. Very little difficulty arises in defining a mathematical function to agree with elevation and slope values at the vertices, and these functions have surface continuity between adjacent triangles. The difficulty comes in obtaining continuity of slope between adjacent triangles at the mid-point of the sides. Further work will undoubted clarify the preferred methods to be used.

One marked difference between triangulation techniques and grids is that smoothing is an inevitable product of gridding, and is normally not a product of triangulation. This is basically an advantage in triangulations, but there is no doubt that where there is an appreciable error component in the x, y or z observations at each data point an unsmoothed map is unattractive to many users. Gold (1979) has used neighbouring data points to estimate a mean and standard deviation at a central data point and from this to perform various types of controlled smoothing and examination of residuals. The presence of a high standard deviation means that the neighbours do not agree well among themselves. This may be due to measurement error, inadequate sampling frequency or real slope discontinuities such as ridges or cliffs.

The Topographic Model

This phrase has been widely, and perhaps loosely, used. Most often it is considered to be a fine grid of elevation data - primarily because that is where most of the computational effort went. What happens in the case of triangulation techniques? As has been mentioned, the cost of triangulation is little more than the cost of reading the data unless manual triangulation is used to define certain features.

In computer terms at least, a topographic model should be distinguished from a topographic map. As with the balsa-wood or styrofoam model, it should be viewable in many ways - from any orientation, by slicing it, etc. A contour map is merely one way of displaying the topographic model. The primary requirement for any display of the model is that it may be interrogated to obtain the elevation at any x-y location, and that this value should be obtained in some reasonably efficient manner. Since there will not usually be a data point precisely at each desired location, a topographic model should be defined as a set of data points plus the required algorithms to obtain any requested elevation.

Gold (In Press) describes problems encountered in obtaining reasonable zero-thickness contours on isopach maps and concludes that, rather than contouring the value "thickness", isopachs should be considered as the difference between two distinct topographic models. He states:

"Two important conclusions may be derived from this example, and in the opinion of this writer they hold true for all map types - whether topographic or thematic. The first is that a model consists of an attempt to generate a space-covering map of a single parameter only. It should conform as closely as possible to the original data and avoid synthetic devices such as grids. The second point is that grids or rasters should be display devices only, to be used as needed to compare and display models. Ideally they should not be preserved or used again to make subsequent comparisons."

Conclusions

Where are we now with triangulation-based terrain modelling? Current techniques are less appropriate for very dense, noisy data than some of the available gridding techniques. However, where data storage is not a problem - either with large computers or smaller data-sets - triangulation techniques can provide a flexibility of use not readily available where grids have to be generated, preserved and compared. It is to be hoped that the next few years will bring further understanding of the potential of relational cartographic data bases of this type.

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