HYPER-ISOMETRY : n DIMENSIONAL MAPPING IN TWO PROJECTIVE SPACES

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ABSTRACT

Mathematical projections have facilitated the expression of complex mapping data in modes convenient to it's distribution and manipulation. The various planimetric displays of geographic data are a common example of this utility. To date, perceptual limitations have restricted most mapping operations to three dimensions. By the sequential application of appropriate transformation equations to the rectangular coordinates of an n dimensional point it is possible to display forms and functions in the two and three dimensional spaces with which we are familiar. The simple relationships between the projection elements and the minimal distortions of the projected figures using the isometric projection make this a suitable schema to discuss here. No attempt will be made to address rotation or translocation of the projected figures; nor will proofs of the operations be discussed, as these considerations are beyond the scope of this paper.

INTRODUCTION

Beginning in 1637 with the <u>Geometry</u> of Descartes, analytic geometry has provided the essential foundation for the mapping of the universe. Descartes limited his discussion to two dimensions; and the technique has since been expanded to visually describe numerical data in three dimensions. However, due to our ability to directly perceive in three dimensions at most (exclusive of our perception of time, in time) three dimensional space has been the effective limit of our mapping abilities.

Various graphic techniques have been developed to allow the display of forms in fewer dimensions than those occupied by the objects displayed. Mathematical projections are an example of such techniques.

This paper will describe the geometry of one such projection. It will illustrate an algorithm for developing isometric transformation equations for mapping n dimensional forms in n-1 spaces. Through a sequential application of these equations it will become possible to graphically display information from any number of dimensional spaces in the two and three dimensional spaces in which cartographers usually communicate.

CONCEPTS AND CONVENTIONS

Isometry and Axonometry

The isometric projections are a special class of those projections known as axonometric. Axonometric projections are distinguished in that all lines of projection are normal to the space upon which the projection is made. Isometric projections , as used here, will be understood as that set of axonometric projections where the direction angles from all of the n space coordinate axes to the lines of projection are equal. Figure 1 illustrates these ideas as applied to projections from three dimensional space into two dimensional space.

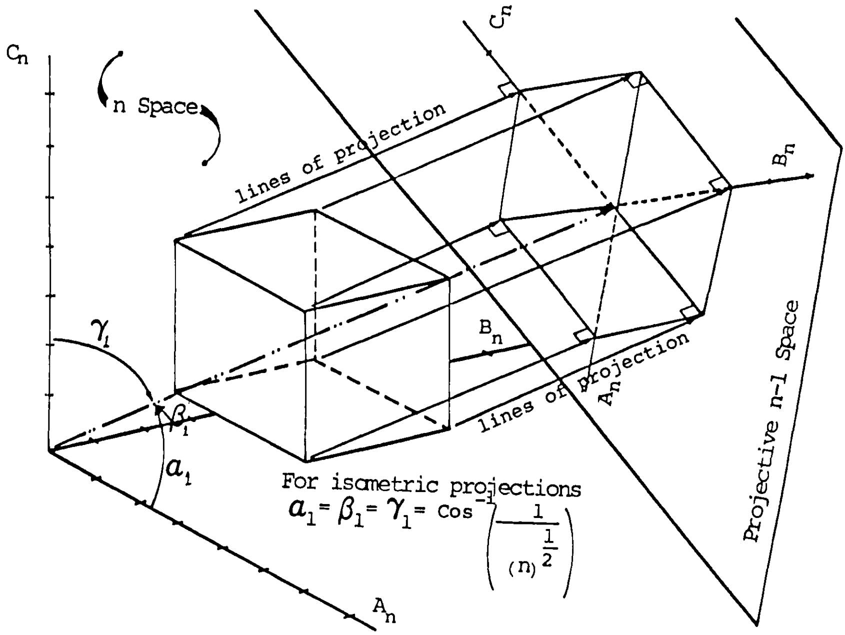

FIGURE 1.

Notation Conventions

Due to the unlimited dimensional spaces to which this process may apply, certain notation conventions for angles, coordinates and axes will be modified. For this discussion the following notation will be observed:

A) The rectangular coordinates of a point will be expressed as (A,B,C,...) rather than as (X,Y,Z,...). In this context the terms A,B,C,... will be refered to as the "coordinate elements." Similarly, the respective axes are termed the A axis, B axis, C axis,... .

B) Points in n space will be denoted as P_n, with coordinates $(A_n, B_n, \ldots)$. Points in n-1 space shall be denoted as P_{n-1}, with coordinates $(A_{n-1}, B_{n-1}, \ldots)$. The coordinates of points P_n and P_{n-1} so expressed shall be known as the "coordinate sets".

C) The direction angles from the axes to any point shall be labled α, β , γ ,...

D) The angular distance between the n projected axes in n-1 space are all equal to each other; the angular

distance will be expressed by the symbol $\aleph$.*
In addition, each of the coordinate, axial and angular expressions will be subscripted to reflect the appropriate dimensional space.

These labeling conventions will ultimately prove inadequate for dimensional spaces such that $n > 24$, and may prove inconvenient where the english and greek symbols are assigned to constants and variables by previous custom. I have adopted this notation only to facilitate the present discussion, and will defer a final resolution till the future.

PROJECTION PROCESS

Coordinate Grids

As all coordinate sets will express the rectangular coordinates of a point, the coordinate axes are perpendicular to each other in their appropriate dimensional space. To project the n axes of the coordinate grid of n space it suffices to generate the median and vertices of the unique equilateral simplex polytope in n-1 space.** The median of this figure represents the origin of the n space grid as projected into n-1 space, and the lines extending from this median through the vertices represent the n projected axes. The negative extension of these axes is the reverse extension through the median from the appropriate vertex. The vertices are situated at a distance, h, from the median. The value of h will be determined below.

The angular distance between each of the projected vertices is determined as follows:

$$\text{(eq. 1)} \quad \aleph_{n-1} = \mathrm{Cos}^{-1}\left(\frac{-1}{(n-1)}\right)$$

This is illustrated with the projection into two dimensions of the three dimensional coordinate grid using the median and vertices of an equilateral triangle (the equilateral simplex polytope in two dimensions), where:

$$\text{(eq. 2)} \quad \aleph_{2} = \mathrm{Cos}^{-1}\left(\frac{-1}{2}\right) = 120^\circ$$

The scale along the projected axes, l_{n-1}, is related to the scale in n space, l_n, by a function of the direction angles to the lines of projection in n space (α_1, β_1,...), where:

$$\text{(eq. 3)} \quad \alpha_1 = \beta_1 = \ldots = \mathrm{Cos}^{-1}\left(\frac{1}{(n)^{\frac{1}{2}}}\right) \text{ ; and}$$

$$\text{(eq. 4)} \quad l_{n-1} = l_n \cdot \mathrm{Sin}\,\alpha_1 = l_n \cdot \mathrm{Sin}\,\beta_1 = \ldots$$

If a true scale is desired for the projected figure in n-1

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* This symbol, especially when subscripted, must not be confused with $\aleph_{\_}$ (Aleph-sub-_ series) which are used in transfinite number theory as the cardinality of infinite sets.
** Simplex Polytopes are the n dimensional figures having n+1 vertices, expressed by the Shlaffli symbol (3,3,...,3). The tetrahedron, a simplex polytope in 3 space, has the Shlaffli symbol(3,3):3 triangular faces at each vertex.

space let $l_{n-1}=1$. The resulting figure is more properly termed an isometric drawing; but involves no additional distortions than those resulting from the projection. Depending on the intended use of the projection either scale may be employed.

Geometric Relationships Between Grids

To facilitate the remainder of the discussion the following assumptions will be made regarding the relationship between the projected n space grid and the rectangular n-1 space grid:

A) The origin of both grids is coincident, and labeled O.

B) The highest order axis on both grids (those having the largest alphabetic expression where $A < B < \ldots < Z$) are coextensive. The highest order axial pair are coplanar. The highest ordered n-1 combinations are cospatial in the appropriate number of spaces. (see Fig. 2)

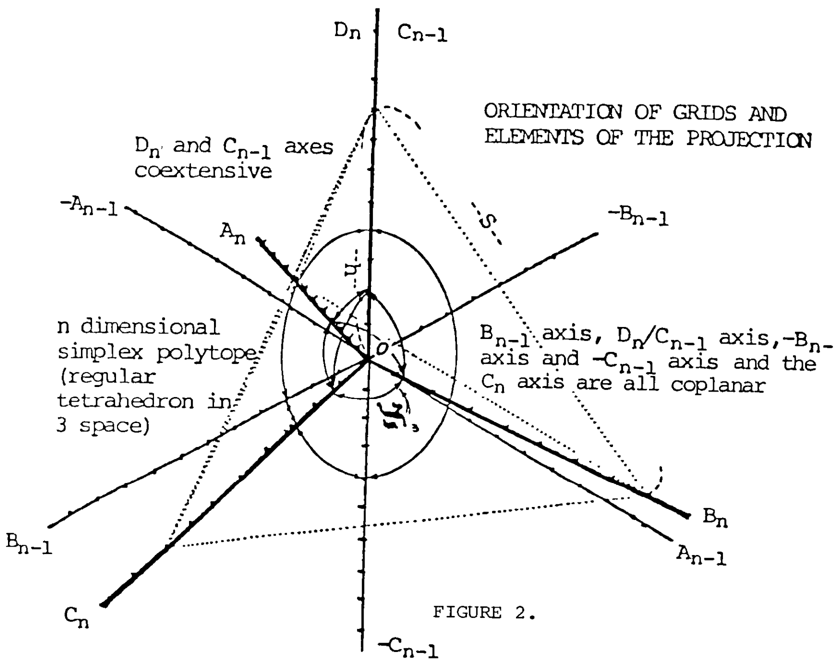

FIGURE 2.

The Projected n Space Grid

The vertices of all equilateral simplex polytopes are equidistant from it's median at a distance, h, which is related to the side length, s, by the following equation:

$$h_n = \frac{(n-1)\cdot(s^2-(h_{n-1})^2)^{\frac{1}{2}}}{n} \qquad \text{(eq. 5)}$$

By assigning a coordinate set (0,0,..,0) in n-1 dimensions to the median and orienting the vertices of the simplex polytope for n-1 space as noted above, the n vertices take the coordinate elements as listed in table 1; and the sum of all like elements must equal 0. Table 1 is calculated assuming s=1.

TABLE 1.*
n-1 Space Coordinate Elements

Vertices of n Space Simplex Polytopes	A_{n-1}	B_{n-1}	C_{n-1}	D_{n-1}	E_{n-1}	...
A_n	-0.500000	-0.288675	-0.204124	-0.158114	-0.129099	
B_n	0.500000	-0.288675	-0.204124	-0.158114	-0.129099	
C_n	0	0.577350	-0.204124	-0.158114	-0.129099	
D_n	0	0	0.612372	-0.158114	-0.129099	
E_n	0	0	0	0.632456	-0.129099	
	$\Sigma=0$	$\Sigma=0$	$\Sigma=0$	$\Sigma=0$		
⋮						

Thus, to calculate the n-1 dimensional coordinates of the n vertices of the respective simplex polytope, oriented as noted, it is sufficient to calculate the value, h. This value is assigned as the highest order coordinate element in n-1 space to the highest order n dimensional axis. The lower order axes are defined by the remaining vertices, which have a value, c, where:

(eq. 6) $$c = -\left(\frac{h}{n-1}\right)$$

The vertices of an equilateral triangle so oriented with s = 1 will then have the coordinates: $\text{Vertex}_n = (A_{n-1}, B_{n-1}, C_{n-1}, \ldots)$.

$A_3 = (-0.50000, -0.2886)$ $B_3 = (0.5000, -0.2886)$

$C_3 = (0, 0.5773)$ See Figure 3. & Table 1.

In order to calculate the direction cosines between any n-1 rectangular coordinate axis and the various projected n dimensional projected axes, plane trigonometry yields:

(eq. 7) $$\cos \alpha_{n-1} = \frac{A_{n-1}}{h};\ \cos \beta_{n-1} = \frac{B_{n-1}}{h};\ \cos \gamma_{n-1} = \frac{C_{n-1}}{h}; \ldots$$

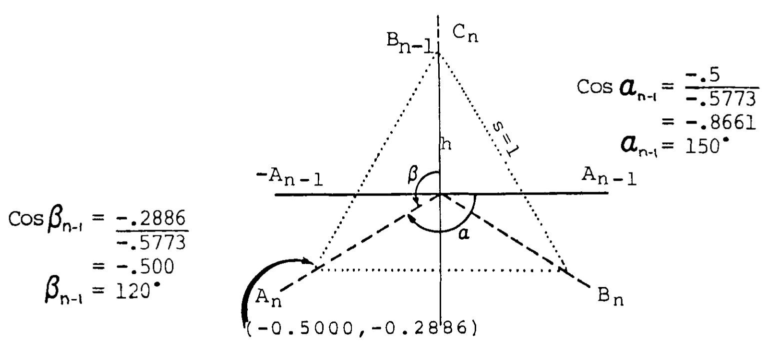

FIGURE 3.

*The n-1 coordinate elements for the n vertices are constructed from the appropriate n-1 x n matrix, from table 1, taking the first n-1 elements in each of the first n rows.

This calculation of direction cosines is repeated until all necessary values are generated. Tables 2,3 and 4 list the values for projections from three space, four space and five space respectively:

TABLE 2.
Direction Angles from 2 Space Axes

Projected 3 Space Axes	α_2	β_2
A_3	150°	240°
B_3	30°	120°
C_3	90°	0°

TABLE 3.
Direction Angles from 3 Space Axes

Projected 4 Space Axes	α_3	β_3	γ_3
A_4	144°.7356103	118°.1255056	109°.4712206
B_4	35°.2643897	118°.1255056	109°.4712206
C_4	90°	19°.4712206	109°.4712206
D_4	90°	90°	0°

TABLE 4.
Direction Angles From 4 Space Axes

Projected 5 Space Axes	α_4	β_4	γ_4	δ_4
A_5	142°.2387561	117°.1573328	108°.8292311	104°.4775112
B_5	37°.7612439	117°.1573328	108°.8292311	104°.4775112
C_5	90°	24°.0948425	108°.8292311	104°.4775112
D_5	90°	90°	14°.4775112	104°.4775112
E_5	90°	90°	90°	0°

Transformation Equations

Since the n coordinate elements of a projected point, P_n, will each be located on it's corresponding axis at a distance from the origin specified by the numerical value of each particular element, these points define the vertices of another simplex polytope in n-1 space. This new simplex polytope will not, in most cases, be equilateral. The n-1 dimensional coordinates of these vertices are a function of the direction angles to the appropriate projected n space axis upon which they are situated.

The projected point P_{n-1} is the median of this second simplex polytope. To calculate the n-1 dimensional coordinates for P_{n-1}, the coordinates for each of the n new vertices is separately determined. Next, the median value for each of the n like elements is calculated, and the resulting set of median values is assembled as the coordinate set of the point P_{n-1}.

(eq. 8) $A_{p_{n-1}} = \frac{\sum_{m=1}^{n} A_m}{n}$; $B_{p_{n-1}} = \frac{\sum_{m=1}^{n} B_m}{n}$; ...

(eq. 9) $P_{n-1} = (A_{p_{n-1}}, B_{p_{n-1}}, C_{p_{n-1}}, \ldots)$

Thus far we have assumed a scale factor between the two grids of 1:1. This assumption allows for the generation of either isometric projections or isometric drawings. The following set of equations combines the operations described in equations (8) and (9); and provides for the application of the desired scale factor, l_{n-1}. The equations are listed for projections from three space into two space, four space into three space and five space into four space. Further levels of projection may be developed from the preceeding operations and observations.

(eq.10) $A_2=l_2\cdot((A_3\cdot\text{Cos } 150°)+(B_3\cdot\text{Cos } 30°))$

(eq.11) $B_2=l_2\cdot(((A_3+B_3)\cdot\text{Cos } 120°)+C_3)$

(eq.12) $A_3=l_3\cdot((A_4\cdot\text{Cos } 144°.7356103)+(B_4\cdot\text{Cos } 35°.2643897))$

(eq.13) $B_3=l_3\cdot(((A_4+B_4)\cdot\text{Cos } 118°.1255056)+(C_4\cdot\text{Cos } 19°.4712206))$

(eq.14) $C_3=l_3\cdot(((A_4+B_4+C_4)\cdot\text{Cos } 109°.4712206)+D_4)$

(eq.15) $A_4=l_4\cdot((A_5\cdot\text{Cos } 142°.2387561)+(B_5\cdot\text{Cos } 37°.7612439))$

(eq.16) $B_4=l_4\cdot(((A_5+B_5)\cdot\text{Cos } 117°.1573328)+(C_5\cdot\text{Cos } 24°.0948425))$

(eq.17) $C_4=l_4\cdot(((A_5+B_5+C_5)\cdot\text{Cos } 108°.8292311)+(D_5\cdot\text{Cos } 14°.4775112))$

(eq.18) $D_4=l_4\cdot(((A_5+B_5+C_5+D_5)\cdot\text{Cos } 104°.4775112)+E_5)$

Order of Processing

As the stated intent of this paper is to introduce a method for generating images in two and three dimensions of forms and functions extant in higher dimensional spaces, the equations developed above must be processed in descending order; from n space into n-1 space, then from n-1 space into n-2 space, etc. Once the projection is made into three dimensions any of the existing methods of projection into two dimensions may be employed. However, a certain consistency of results is lost if the projection schema is changed.

APPLICATIONS

As with any mapping operation, it is the responsibility of the cartographer to determine the appropriate projection method for the display of the information. To date, the options for mapping higher dimensional spaces has been quite limited. Hopefully, this is now a temporary state of affairs.

A complete list of possible applications for this type of projection is unnecessary; indeed, it would be impossible to delimit. However a suggestion of the potential for such a system may be gleaned from the following discussion of the relativistic curvature of the universe; "The analogy," visualizing the two dimensional space of a plane curving around the surface of a sphere,"collapses because it is hopeless to imagine what the extra spatial dimension looks like. No one has ever seen it." Callahan (1976) We may still be unable to see in these spaces; but we may now visualize constructs in them, and we may manipulate those constructs to our purposes.

The following set of illustrations are various regular n dimensional polytopes from the indicated number of dimensions projected using the above described processes.

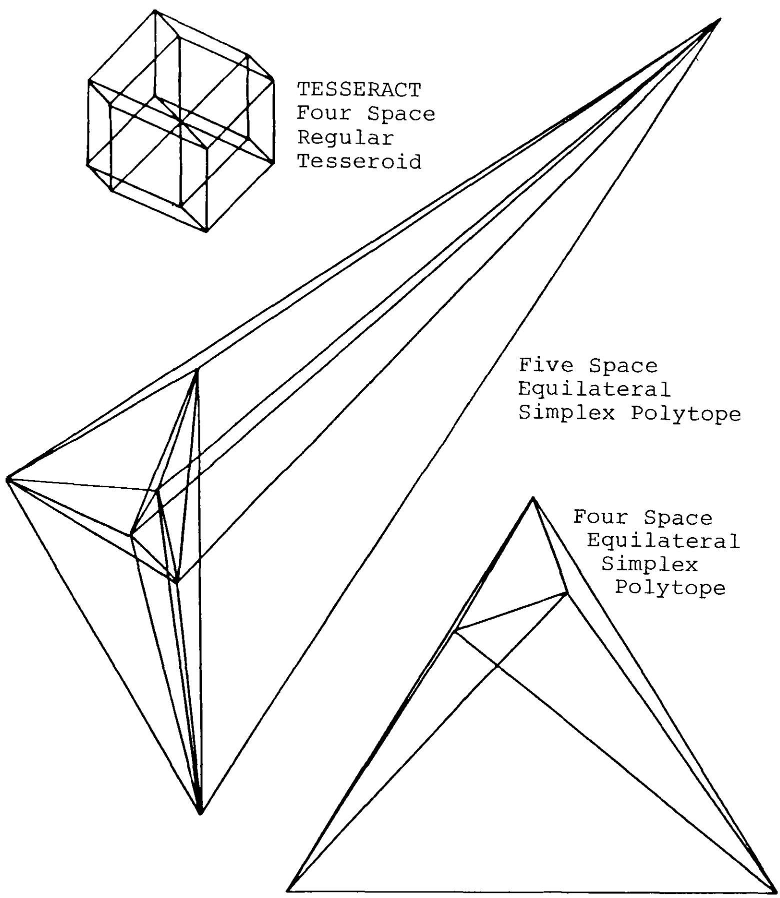

FIGURE 4.

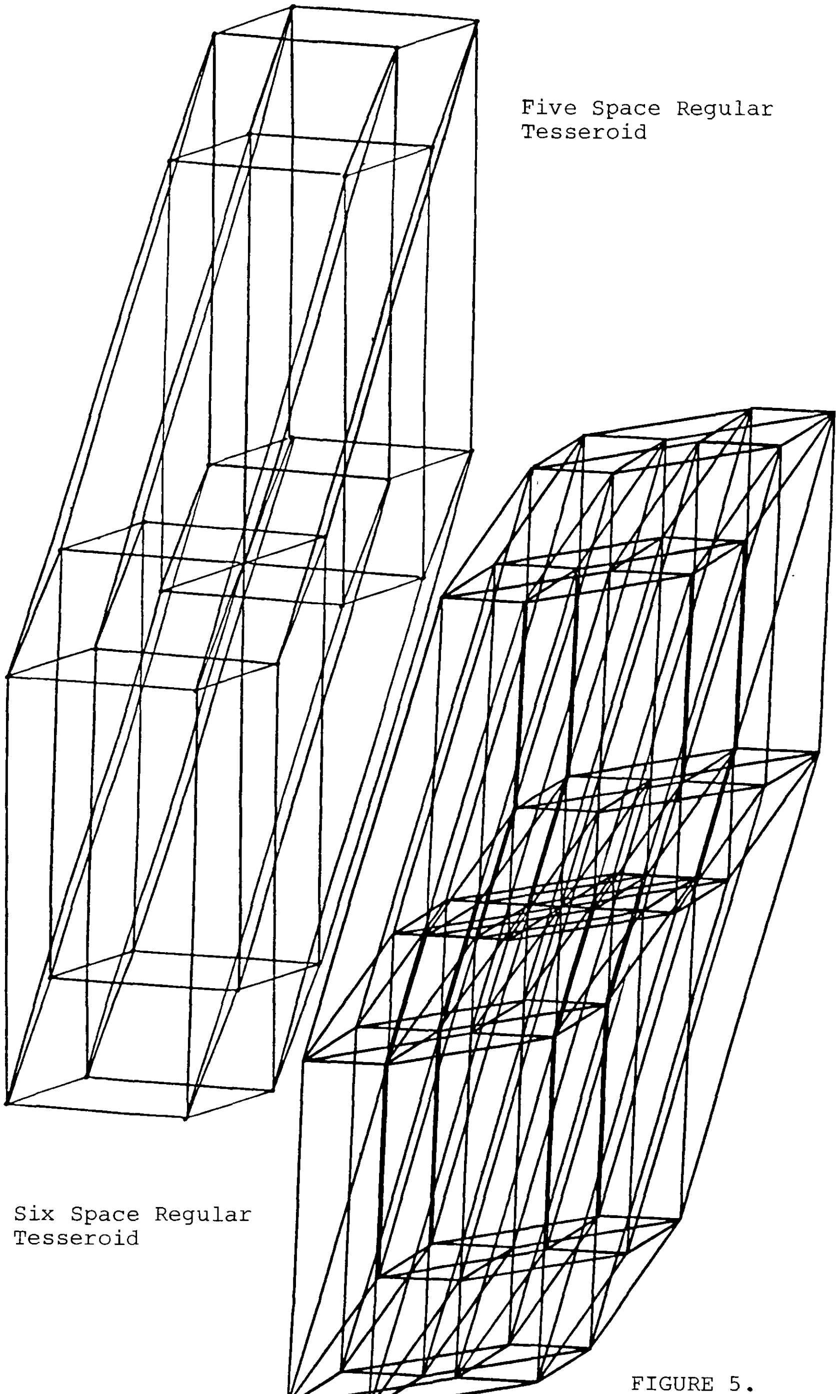

FIGURE 5.

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