

TRIANGULATIONS FOR RUBBER-SHEETING

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ABSTRACT

This paper focuses on the application of triangulation and rubber-sheeting techniques to the problem of merging two digitized map files. The Census Bureau is currently developing a map merging procedure called conflation. Reproducibility, quality control, and a desire for mathematical consistency in conflation lead to a need for well-defined procedures. The Delaunay triangulation is well-defined and in some sense the 'best' triangulation on a finite set of points. It leads naturally into an efficient rubber-sheeting algorithm. The discussion starts with triangulations and rubber-sheeting in general and well-defined triangulations. This leads to the Delaunay triangulation, an algorithm for that triangulation and a specific rubber-sheeting technique. Finally, some problems that require further research are mentioned in an appendix.

INTRODUCTION

The Statistical Research Division of the Census Bureau is currently applying rubber-sheeting techniques in developing a system for merging two digitized map files of the same region. Our work began with an evaluation of algorithms for triangulating plane regions. Our needs led us to choose the Delaunay triangulation and to design and implement new algorithms for its maintenance and manipulation. This particular triangulation permits specialized operations that are not possible with other triangulations. This paper details the algorithms we have developed, describes the underlying theory used, and outlines some ongoing research.

The first section describes rubber-sheeting and triangulations in general and how we employ these techniques. Rubber-sheeting, which in our case is known technically as Piecewise Linear Homeomorphism (PLH), is used for transforming the coordinates of digitized map files. The PLH is a mathematical transformation that preserves topology and is linear on subsets of the map. We use triangulations to divide the map into the subsets on which the linear transformations are constructed.

The second section explains the need for a well-defined triangulation and discusses the consequences, i.e., uniqueness, and some of the specialized routines that are possible with such a process. There are two principal benefits arising from a well-defined triangulation. The first is the ability to exploit the underlying mathematical theory used in defining the triangulation. The other is the security of knowing the output will be unique. The specialized routines arise as consequences of the mathematical theory. These routines include local operations that are globally consistent and correct.

The third section defines the Delaunay triangulation and describes the underlying mathematical theory. This triangulation is well-defined and is in some sense the 'best' triangulation on a set of vertices. We present in this section several equivalent characterizations of the Delaunay triangulation, focusing on local properties.

The fourth section contains a detailed description of the triangle determination routines that have been developed for implementing the theory. This section also contains a description of the data structures we used.

The final section describes the rubber sheeting transformations which operate on the triangles of the Delaunay triangulation. These routines use the fact that a triangle is an elementary simplex and efficiently generate the convex coefficients for any point inside.

An appendix addresses two areas for continuing research. These areas are 1) whether a triangulation can be extended in a topologically consistent manner to another set of vertices, and 2) how one handles the case of four or more co-circular vertices.

SECTION 1. TRIANGULATION AND RUBBER-SHEETING PRELIMINARIES.

There are many instances in cartography where dividing a region into smaller subregions is beneficial. One useful technique for subdividing a region is triangulation. At the Census Bureau, triangulations are used to define rubber-sheet transformations which are then employed in the process of merging digitized map files.

First, it is necessary to define the term triangulation. Consider $N \geq 3$ points in the plane which are not all colinear. Let R be the region bounded by the smallest simple convex polygon containing the N points. A triangulation is a maximal subdivision of R into triangles where the N points are the vertices. Every point in R is contained in one and only one triangle, except if the point lies on a triangle edge then it may be in more than one. (See figure 1) So, a triangulation may be viewed as a jigsaw puzzle where each piece is a triangle, and, using only the N points as vertices, no triangle may be divided into smaller triangles. The picture that is made by piecing the puzzle together is that of the region R . Note that for five or more points in the plane there is more than one way to triangulate these points. Also, every triangulation on a given set of points has the same number of triangles. (See Lee and Schachter, 1980)

A rubber-sheet transformation is a mathematically defined function from one region to another both of which have been divided into subregions. Each subregion in the first has a unique counterpart in the second. Also, every point and line of the boundaries and the subregions themselves maintain their relative positions with one another from the first region to the second, i.e., topology is preserved. The transformation is specified in pieces—a specific subtransformation for each subregion. Thus, the name "rubber-sheet" should be clear. Each subregion is transformed differently, much like taking a piece of rubber and stretching it in sections to make it fit over some object (see White and Griffin, 1985).

The specific rubber-sheet transformation that we use is called piece-wise linear homeomorphism, or PLH. The 'pieces', subregions, that the PLH is defined over are the triangles from a triangulation. It is quite easy then, to define a linear transformation from a triangle on one map to its counterpart on the other. Each linear transformation is a homeomorphism, i.e. it preserves topology. Combining this with the fact that topology is preserved between the triangulations, the resulting composite function is piece-wise defined, linear on each piece, and a homeomorphism over the entire map. Thus, we get a PLH.

* A simple convex polygon is a polygon where the edges meet only at the vertices and all exterior angles at the vertices are $\geq 180^\circ$.

The procedure being developed for merging two digitized map files is called conflation. In part, it is an iterative process, each loop consisting of the following steps: 1) selection of pairs of vertices, one from each map; 2) triangulating the vertices of one (rubber-sheet) map; 3) transferring the triangulation to the vertices of the second (stable base) map; and 4) generating and performing a PLH transformation on the nodes of the rubber-sheet map. The result of a loop is that the features of the rubber-sheet map are in closer alignment with those of the stable base map. In addition, the node pairs which were selected as vertices are made to coincide.

SECTION 2. WELL-DEFINED TRIANGULATION.

There are many ways to triangulate a set of points in the plane and some ways depend on the ordering of the points. As a result of there being options, some problems arise. Use of a well-defined triangulation and procedure will eliminate the problems and also gives rise to other benefits.

A well-defined triangulation is a set of properties or rules used for building triangulations that has a mathematically precise formulation and does not depend on the order in which points are processed. These properties allow us to determine which vertices are connected by edges. The effect is the output from a well-defined triangulation procedure is unique. For example, take the triangulation defined by the rule that the total length of all the edges is minimum. This is well-defined, it doesn't matter what order points are processed. The minimum length of edges is always the same for the entire set of points. Note the two uses of the word 'triangulation'. Earlier we meant the set of triangles created on the points. Now we also use it to refer to the properties or rules used to build the triangles. The context will make the meaning clear.

The definition above is one that is global, i.e. we ask whether or not the resulting triangulation is unique. We can also look at the problem from another point of view. A locally well-defined triangulation has the property that given any three vertices it can be determined whether they form a triangle without processing the entire triangulation. Since the local condition clearly implies the global one, it is convenient to work with locally well-defined triangulations. An example of such a triangulation, the Delaunay triangulation, will be described in the next section.

The problem associated with the use of ambiguously defined triangulations is that the output may not be unique. The output depends on the order the data are processed. This has ramifications in a process such as conflation. Maps are merged using rubber-sheet transformations which are first defined by triangulations. A change in a triangulation will result in a change in the rubber-sheet transformation. This might result in different pairs of features being identified. Thus, quality control and reproducibility are very difficult if not impossible to manage.

We can see the problem in another way by looking at a simple example. Suppose we start with a square divided into two triangles. For each additional vertex, find the triangle containing the point and create three new triangles formed by the new vertex and each of the three pairs of vertices from the old containing triangle. (See figure 2) It is easy to see that the resulting triangulation (after just two new points are added) depends upon the order in which data are processed. (See figure 3)

The local property allows us to create some specialized routines that otherwise would not be possible. There are many such routines that could be developed. Here we will briefly describe two of them: DELETE and LOCATE.

DELETE is used to update a triangulation after removal of a vertex. Clearly, such a routine can be built for any triangulation procedure, however, the output may not be unique. Here, the local characterization is used to build the new triangles inside the simple polygon created by removal of the vertex. Therefore, the new triangles will be defined in the same way as the rest of the triangulation. Moreover, the resulting triangulation will be the same as that created from scratch given the modified set of vertices.

LOCATE builds the triangle surrounding a given point without building the entire triangulation. An iterative procedure produces triples of vertices which are tested. The local definition determines whether the three vertices form a triangle. If so, it is easy to test whether the point lies in the triangle. If it does not, the procedure picks another triple in the appropriate direction. The triangle that is constructed at the end is the same as the one found to contain the point from the entire triangulation. This has to be the case, due to the local property.

SECTION 3. THE DELAUNAY TRIANGULATION.

The Delaunay triangulation is a globally and locally well-defined triangulation. It has properties which make it particularly useful for conflation. We give five equivalent definitions and properties of the Delaunay triangulation. For a more detailed discussion of the definitions, see Lee and Schachter, 1980.

Before we define the Delaunay triangulation, we need to restrict the set of vertices slightly. Assume that no four vertices are co-circular. The reason for this will become clear shortly.

We also need to define the Thiessen polygons on the set of vertices. Imagine that around each vertex a circle grows out like a wave, all at the same rate, starting at the same time. As a wave grows out, it freezes at all points of contact with other waves. The effects are that there is a unique simple convex polygon around each vertex which is not on the boundary of a triangulation and a unique unbounded polygonal region for each boundary vertex. These polygons are the Thiessen polygons. (See figure 4)

The following equivalent statements each characterize the Delaunay triangulation. The proof of their equivalence is in Lee and Schachter.

- (local) 1) Three vertices form a triangle if and only if the circumcircle of these vertices contains no other vertex.
- (local) 2) Two vertices are connected by an edge of the triangulation if and only if there exists a circle containing the vertices and no other vertex.
- (global) 3) Given any triangulation on the set of vertices, change it in the following way. Consider each pair of triangles that shares an edge. Determine whether the circumcircle of one of the triangles contains the vertex not shared by the other. If so, then consider the pair of triangles as a quadrilateral. Replace the diagonal (the common edge) with the other, forming two different triangles. Continue until no more swaps can be made. (The updating criterion is called the local optimization procedure, LOP, in Lee and Schachter, 1980).
- (global) 4) The triangulation formed by drawing an edge between any two vertices whose Thiessen polygons share an edge. (See figure 4)
- (global) 5) Maximize the minimum measure of angles of all the triangles.

SECTION 4. OUTLINED ALGORITHM OF DELAUNAY TRIANGULATION.

PRELIMINARIES

A) Lines: Each line is expressed as $\{ (x, y) \mid Ay + Bx + C = 0 \}$ where the line passes through the points (x_1, y_1) , (x_2, y_2) and $A = x_2 - x_1$, $B = y_1 - y_2$, $C = y_2 x_1 - x_2 y_1$.

B) Arrays:

- 1) VERTEX = stores pointers to each vertex in counter-clockwise order for each triangle
- 2) NABORS = stores pointers to each neighbor in counter-clockwise order for each triangle
- 3) EDGES = stores A,B,C for each line of each triangle
- 4) CIRCLE = stores x,y coordinates of center and square of radius of circumcircle for each triangle

Note: For all the arrays, the i-th record points to the i-th triangle.

ALGORITHM

- I) Initialize = start with 4 corner points each 10% beyond N&E, S&E, S&W, and N&W boundaries of map respectively. Take 1st vertex from map and create 4 triangles. Initialize VERTEX, NABORS, EDGES, and CIRCLE arrays with data from these triangles.
- II) For each vertex (NEW)
 - A) Find triangle containing NEW (1st is found if point is on boundary). For each triangle constructed so far
 - 1) Perform LOP* on triangle and the point NEW
 - 2) Pass - test each line to see if NEW lies in positive half-plane
 - a) All positive, found triangle, go to II-B
 - b) First negative, neighbor of this line is next triangle, go to II-A-1
 - 3) Fail - go to II-A
 - B) For each new triangle (3 new ones if from II-A-2-a, 2 new ones if from II-C-2-b)
Update VERTEX, NABORS, EDGES, CIRCLE
 - C) For each adjacent pair of triangles, only one of which has NEW as a vertex
 - 1) Perform LOP on triangle of pair without NEW with the point NEW
 - 2) Pass
 - a) Switch diagonals of quadrilateral formed by triangle pair
 - b) go to II-B
 - 3) Fail - go to II-C

* See Lee and Schachter, 1980, for details about the subroutine LOP.

SECTION 5. THE PLH RUBBER-SHEETING TRANSFORMATION.

The rubber-sheeting transformation, a PLH, is particularly easy to compute. The method and underlying theory will be presented as a list of facts below. For details, see Saalfeld, 1985. Keep in mind that a separate linear transformation is defined for each triangle.

- 1) A triangle is an elementary simplex, i.e., any point p inside a triangle can be expressed uniquely as $p = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$ where $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\alpha_i \geq 0$, and V_i are the vertices of the triangle. The α_i are called the convex coefficients of p .
- 2) A line in the plane passing through two points (x_1, y_1) and (x_2, y_2) can be expressed as $L = \{ (x, y) \mid Ay + Bx + C = 0 \}$ where $A = x_2 - x_1$, $B = y_1 - y_2$, $C = y_2 x_1 - x_2 y_1$.
- 3) Each line divides the plane into two half-planes:
 $L^+ = \{ (x, y) \mid Ay + Bx + C \geq 0 \}$ and
 $L^- = \{ (x, y) \mid Ay + Bx + C \leq 0 \}$.
- 4) If the vertices of a triangle are ordered counter clockwise, then the interior of the triangle is $L_1 \cap L_2 \cap L_3$ where L_i is the line on which the i -th edge lies.
- 5) Let L_i be the line of the edge opposite vertex $V_i = (x_i, y_i)$. If the formula for L_i is $Ay + Bx + C$, then the area of the triangle $A_T = \frac{1}{2} \mid A y_i + B x_i + C \mid$.
- 6) For any point p in a triangle T , the i -th convex coefficient for p is $\alpha_i = A_{pi}/A_T$ where A_{pi} is the area of the triangle formed using the i -th edge and p .
- 7) If the vertices are in counter clockwise order, the absolute value sign in 5 is not necessary. Moreover, for any point p and any triangle T , A_{pi} is non-negative for all i if and only if T contains p .
- 8) Let p be a point in a triangle, T . Let T' be the triangle corresponding to T in the other map. Let $\alpha_1, \alpha_2, \alpha_3$ be the convex coefficients of p in T . Let V'_1, V'_2, V'_3 be the vertices of T' . Then the image of p under the PLH is $p' = \alpha_1 V'_1 + \alpha_2 V'_2 + \alpha_3 V'_3$.

CONCLUSION

Of the different kinds of triangulations on a finite set of points, well-defined triangulations are best suited for applications such as conflation. These triangulations enable the user to build specialized subroutines that operate locally but are globally consistent. The well-defined Delaunay triangulation is easily described and implemented. Moreover, the implementation requires ideas that can be used to define a rubber-sheet transformation. The Delaunay

triangulation and its associated rubber-sheet transformation have been implemented for use in the conflation process at the Census Bureau.

APPENDIX. - CONTINUING RESEARCH.

There are a number of issues that require further research. Some of these problems are data structure problems and others are mathematical problems related to the algorithms. Two of the mathematical problems will be discussed here.

The easier of the two is the problem of what should be done about the case of four or more cocircular points. Recall, the definition of the Delaunay triangulation specifies that no four points are cocircular. If four points are cocircular, then there is a choice as to which pair of triangles to form. Unfortunately, the choice is arbitrary; the theory will not allow a resolution of the problem. Allowing software to make an arbitrary choice is unsatisfactory since we are motivated by the desire to have a well-defined triangulation. So, a well-defined method for handling this case is needed to make the theory complete.

The harder question is the extensionality problem. During conflation the triangulation created for one map is extended to the second via the one to one correspondence that exists between the two sets of vertices. If the correspondence lifts to a homeomorphism between the triangulations, then there is no problem. However, this is by no means guaranteed. It is possible, and it happens in practice, that the vertices shift with respect to each other enough to cause triangles to reverse orientation or flip on top of each other. (See figure 5) Topology does not remain consistent in either case. We have implemented an algorithm that recognizes when topology is not preserved, but the crucial question before us is how to fix the triangulation so as to maintain topological consistency and remain faithful to the definition of the triangulation.

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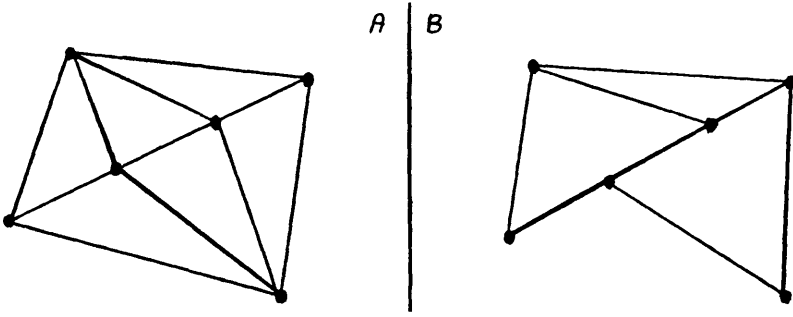


Figure 1. A) Maximal Set of Triangles; B) Non-maximal Set

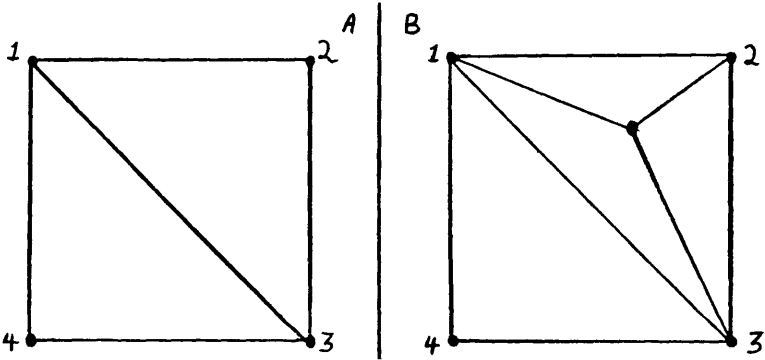


Figure 2. A Simple Ambiguous Triangulation Procedure, A) Before, and B) After Adding a Vertex.

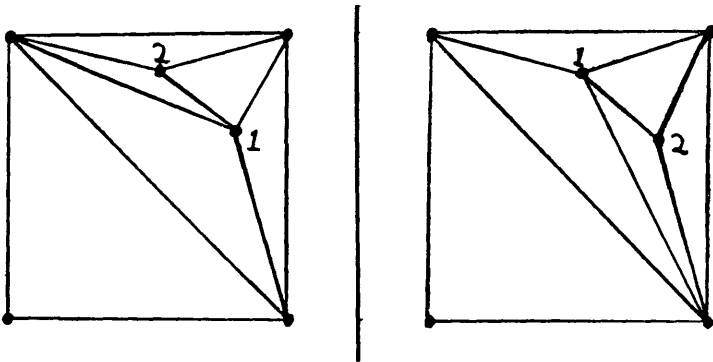


Figure 3. An Example of Order Dependence Using the Procedure of Figure 2.

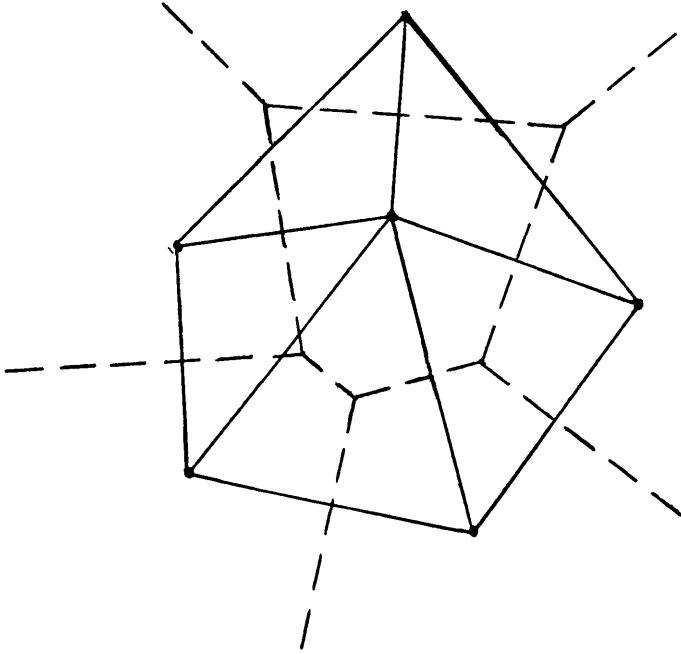


Figure 4. Thiessen Polygons on 6 Points (dashed lines) and the Resulting Delaunay Triangulation (solid lines).

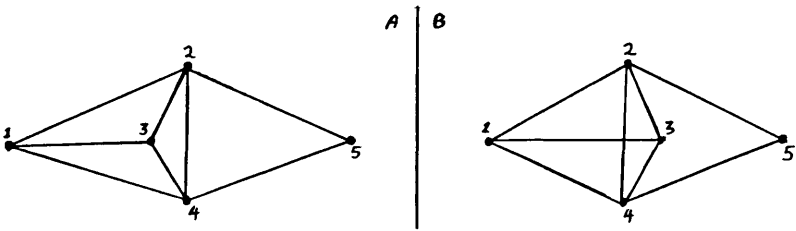


Figure 5. Topological Inconsistency in Extending Triangulation From One Set of Points (A) to a Second (B).