RECURSIVE APPROXIMATION OF TOPOGRAPHIC DATA USING QUADTREES AND ORTHOGONAL POLYNOMIALS*

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ABSTRACT

Orthogonal polynomials (OP) are used to estimate polynomial coefficients and root-mean-square deviations (RMSD) for gridded elevation data within quadtree subquadrants. These subquadrants are recursively subdivided into four if the RMSD exceeds some threshold. Polynomials of orders one through six are fitted to three 256 by 256 DEMs, using RMSD thresholds of 1, 3.5, and 7 meters. The OP-quadtrees required from 9 to 20 percent of original grid space when the RMSD was set at 7 meters, but between 48 and 99 percent of that space for an RMSD of 1 meter. For a fixed RMSD, the total space required appears to be independent of polynomial order. If this effect is true in general, the obvious implication is that order does not matter. In that case, low-order polynomials could be used, saving computation time. When order is held constant, the space required by the OP-quadtree appears to be an inverse power function of the RMSD criterion.

INTRODUCTION

A **digital elevation model** (DEM) can be defined as any machine-readable representation of topographic elevation data. A major issue in DEM research is the selection of an appropriate **data structure** (Mark, 1979). The most frequently-used data structure for DEMs is a regular square grid. One weakness of the grid data structure is its inherent redundancy, and the large amount of computer resources needed to achieve a given accuracy. The grid size must be sufficiently small to capture the smallest feature of interest in the entire study area, and to define the boundaries of larger features to some required level of precision. This implies that cells in most of the region will be smaller than needed; in other words, there will be too many cells.

In an attempt to address this problem, alternative data structures for DEMs have been designed. The most widely used of these is the triangulated irregular network (TIN),

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which represents the terrain by a triangulation on a set of points chosen to represent the surface (Peucker and others, 1978). Whereas TIN are being adopted as a DEM data structure within several of the leading commercial Geographic Information Systems (GIS), the TIN structure is at a disadvantage in terms of data collection. This is because devices are now available to produce very dense regular grids directly from aerial photographs.

At least two strategies for the more direct compression of grid DEMs have been developed. In one approach, grids of varying spatial resolution are employed; in the other, square or rectangular patches of a fixed size are approximated by polynomials or other mathematical functions requiring fewer coefficients than there were grid points. Quadtrees are a spatial data-structure which provides a convenient basis for handling variable resolution data, and allows the variable-resolution and polynomial-patch approaches mentioned above to be combined.

In the present study, orthogonal polynomials are used as an efficient way to estimate polynomial coefficients for gridded data; we call the result the OP-quadtree of the DEM. Polynomials of orders one through six were fitted to each of three 256 by 256 DEMs, using root-mean-squared deviation (RMSD) thresholds of 1, 3.5, and 7 meters. Also, integer RMSDs from 1 to 10 were evaluated for 3rd-order polynomials for the three study areas. Empirical results are presnted, and the implications of these results are discussed.

BACKGROUND

Polynomial Patch Approximations of Topographic Surfaces.

Junkins, Jancaitis, and co-workers applied polynomial patch approximations to DEM surfaces (Junkins and others, 1972; Jancaitis, 1977). They were primarily concerned with the handling "noisy" data from the U. S. Army's UNAMACE image correlation system. Their approach was to divide the surface into small square patches, and to fit low-order polynomials (quadric surfaces, $z = a + bx + cy + dxy$) to the elevations within these patches. The patches were small, and the resulting quadric surface was used regardless of how large the residual variance was. (For noisy data, large residuals are presumed to represent errors which should be removed from the data.) Because this method produces discontinuities along patch boundaries, Jancaitis' group used a weighting function approach to blend together adjacent patches, eliminating undesireable breaks in decompressed data.

Quadtrees.

The **quadtree** is a data structure which is based on a regular decomposition of a square image into quadrants and subquadrants. Basically, the quadtree can be constructed recursively, with a "stopping criterion" which indicates whether a subquadrant should become a terminal (leaf) node in the quadtree, or should be subdivided. In most quadtree research, the stopping criterion is uniformity, that is, a subquadrant is subdivided if it contains any variation.

There is a major problem with the strict application of quadtree concepts to topographic data: it is unusual to find sets of four mutually-adjacent cells which are of identical height. In order to be space-efficient, the quadtree concept must be adjusted. While the recursive spatial structure of the quadtree is retained, the stopping criterion can be modified to include surface approximation within quadrants (Martin, 1982; Chen and Tobler, 1986). A mathematical function is fitted to the heights within a square subquadrant. Then, whenever the RMSD for the elevations within a subquadrant is larger than some predetermined criterion, the procedure is applied recursively to each subquadrant of the current square.

Quadtree-based Surface Approximation.
Martin (1982) used a quadtree-based method for polynomial approximation of DEM data. His procedure fitted a linear equation ($z = a + bx + cy$) to all elevations within a valid quadtree subquadrant. If the RMSD was less than some threshold, the 3 coefficients of the plane were used to represent elevations within the subquadrant. Otherwise, the subquadrant was split into its 4 children, and the procedure was recursively applied to the children. Clearly, the depth of quadtree subdivision, and thus the quantity of data to be stored, will increase if a low threshold is chosen for the RMSD.

In a similar study, Chen and Tobler (1986) fitted five mathematical functions to quadtree subquadrants. The functions chosen were: (1) the mean surface (equivalent to a least-squares polynomial of order zero); (2) a maximum surface (highest elevation in the quadrant); (3) a minimum surface; (4) a "ruled surface," a hyperbolic paraboloid of the form: $z=a_{00}+a_{10}x+a_{01}y+a_{11}xy$; and (5) a quadric surface of the form $z=a_{00}+a_{11}xy+a_{20}x^2+a_{02}y^2$. The coefficients of functions (4) and (5) were determined by substituting into the equation the coordinates of the four corners of the quadrant. Thus the equations pass through the corners exactly, and are not influenced by other cells in the quadrant. Chen and Tobler evaluated goodness of fit according to maximum absolute deviation, rather than RMSD.

The advantage of their approach over a least-squares or orthogonal polynomial method employed in this paper is largely computational efficiency. Furthermore, since the values stored to represent the surface are just elevations, they can be represented using two bytes each; in contrast, polynomial coefficients must generally be represented by floating-point numbers, needing at least 4 bytes each. Chen and Tobler (1986) computed space requirements and running times for two topographic samples, each a 128 by 128 grid of 50 meter cells; they did not discuss the source of their DEM data. Each surface was tested with maximum-error tolerances of 2, 6, and 10 meters. Their test program proceeded recursively, and counted quadtree leaves. The number of leaves was then multiplied by 2 bytes per coefficient plus 2 bytes for the location key (a total of 4 bytes per leaf for functions 1, 2, and 3, and 10 bytes per leaf for 4 and 5). Chen and Tobler found that the 2-meter tolerance produced quadtrees requiring more space than the original grid (2 bytes per elevation) for every function and for each of the topographic samples. For the 10-meter tolerance, the "ruled surface" quadtrees required from 50.7% to 84.4% for the more smooth terrain sample, and from 53.3% to 127.9% for a more rugged area. For every combination of tolerance and topography, the ruled surface required the least space of all functions tested.

Orthogonal Polynomials.
When data are acquired at equally-spaced intervals, orthogonal polynomials provide a computationally efficient way of calculating the coefficients of a polynomial (trend surface) function. Their use in determining least squares coefficients requires substantially less computation because matrix inversion is not required as with traditional regression analysis. As a result, these polynomials have long appeared in **trend surface analysis** (Simpson, 1954; Grant, 1957; for an overview, see Krumbein and Graybill, 1965). One interesting property is that when variables of higher degrees are added to the function, low-order components do not have to be recalculated because the polynomials are independent (hence the term "orthogonal"). For an interesting description of the use of orthogonal polynomials in the one-dimensional case, see Fisher (1973) or Krumbein and Graybill (1965).

For a given sample size N, an N by N matrix, hereafter referred to as the "orthogonal matrix", can be constructed with independent columns that represent individual orthogonal polynomials. If these columns are numbered 0 to N-1, then column i contains the values of the orthogonal polynomial for determining the coefficients of the trend surface variables for degree (exponent) i. Thus, column 0, which is always the unit vector, would be used to calculate the value for a zero-order polynomial equation, resulting in the mean of the dependent variable. In addition, the orthogonal polynomial of column 1 would be used to determine the linear trend in the data; column 2 would be used for the quadratic component. In the same manner, the coefficients for the trend surface function can be computed up to degree N, when a perfect fit is made between the data and the regressive model. DeLury provides a table of these orthogonal matrices up to N=26 (DeLury, 1950). In addition, DeLury also provides a method for generating the orthogonal matrices for larger values of N, but the integral values soon become too large to handle using standard variable types on many computers.

Orthogonal polynomials also can be applied to two-dimensional data. With two dimensions, an N by M data matrix is pre- and post-multiplied by the orthogonal matrices for sample size N and M respectively, thereby allowing the method to be applied to a rectangular grid. However, in the present case, N and M will always be equal, and will be 2 raised to the power of the level. Each value in the resultant matrix, henceforth referred to as a **G**, is divided by the product of the total sum of squares (SOS) for the two orthogonal polynomials that correspond to the value's position in the matrix, producing a new matrix, **B**.

The matrix **B** contains the individual b_{ij} coefficients for the trend surface function, with the position in the matrix indicating the appropriate independent variable; the row represents the exponent of the horizontal (X) component and the column the exponent of the vertical (Y) component. For example, the coefficient for position row=2, column=3 in the **B** matrix would be associated with the independent variable X^2Y^3 of the trend surface equation. It should be noted that the b_{ij} coefficients cannot be interpreted as the marginal effect on the dependent variable as is the case with the traditional b_r regression coefficients. However, the b_{ij} coefficients can be used to determine the appropriate order of the polynomial function, and can be used to generate the best approximate surface for the original elevation data. Krumbein and Graybill (1965) provide an excellent procedural description of the utilization of orthogonal polynomials for a two-dimensional trend surface analysis.

In this study, the criterion for deciding whether a specific polynomial function fits a matrix of data will be based on the square root of the mean square deviation (RMSD) between the polynomial surface and the original data. Computation of the RMSD requires the calculation of matrix **Z** which contains the corrected sum of squares associated with each independent variable. With orthogonal polynomials, the calculation of this matrix is straightforward. First, each element of matrix **G** is squared. Then each value is divided by the product of the SOS of the appropriate orthogonal polynomials in the manner described above. Each value within the resultant **Z** matrix is the corrected sum of squares attributed to an independent variable. Again, the row and column position of a value within the matrix indicates the appropriate independent variable. Given the order of a trend surface function, the sum of all values in the **Z** matrix whose row and column sum is less than or equal to this order constitutes the corrected SOS attributed to the trend surface equation. The sum of the excluded values reveals the residual SOS, which is divided by the sample size and raised to the 1/2 power to determine the RMSD.

METHODS

As noted above, application of the method of orthogonal polynomials to a grid of N by N cells produces a matrix of N^2 coefficients from which the surface can be recreated perfectly (except for truncation and roundoff errors resulting from machine representations). The quadtree-based algorithm computes the orthogonal polynomial coefficients (OPC) for a quadrant, and then uses a decision rule to determine whether the current quadrant should be saved or whether it should be recursively subdivided.

Various strategies for selecting a subset of the OPC, and for assessing whether the resulting fit is 'adequate', lead to different algorithms. In the current study, we used a fixed polynomial order for each run. Because of limitations on computer time and space, the orthogonal matrix for N=32 could not be generated; thus, in our experiments, the quadtree subdivision begins with level-4 nodes (16 by 16 subgrids). Then, the SOS terms associated with all OPC with exponents less than or equal to the polynomial order are added together and used to compute residual RMSD. If this RMSD is greater than the specified maximum allowable error, the data matrix is divided into four subquadrants, and the fitting procedure is applied recursively to each of these. The program does not subdivide the data matrix if the RMSD is less that the threshold, or if the data matrix is too small to calculate a surface function of the specified order.

As noted above, a polynomial trend surface of order O is the sum of polynomial terms of the form $a_{ij}x^i y^j$, such that $i+j \leq O$. It can easily be shown that the total number of terms in such a polynomial is equal to $[(O+1)(O+2)]/2$. This is the number of degrees of freedom available for least-squares estimation of the trend surface, and also represents the minimum number of observations needed to compute the polynomial. However, in order to use the method of orthogonal polynomials, a matrix of size (O+1) by (O+1) must be computed, requiring a data grid of side-length O+1. This places a more severe restriction on the possible polynomial orders which can be computed for small matrices: the maximum order is one less than the side length of the data square. Table 1 relates quadtree levels, space requirements, and possible polynomial orders.

TABLE 1: QUADTREES PROPERTIES AND ORTHOGONAL POLYNOMIAL STATISTICS

1) quadtree properties:

quadrant level	0	1	2	3	4	5
pixels	1	4	16	64	256	1024
bytes	2	8	32	128	512	2048

2) orthogonal polynomial properties:

order	1	2	3	4	5	6
OP terms	3	6	10	15	21	28
bytes for OPC*	14	26	42	62	86	114
minimum level**	2	2	3	3	3	3

* calculations assume that each coefficient is stored as a floating-point number requiring 4 bytes, and add 2 additional bytes for the location key

** minimum level for which OPC require fewer bytes of storage than the original integer grid

We noted above that Chen and Tobler (1986) counted the total number of quadtree leaves of any size (level) needed to represent the surface to some pre-determined accuracy, and then multiplied that number by the number of bytes needed to represent one quadrant of the current equation. For some surfaces and tolerance values, Chen and Tobler found that some quadtrees required more storage space than the original grids. However, it seems more appropriate to store a hybrid structure, counting a leaf-node only when the surface function requires less space than would the original grid over the same quadrant. In the present study, we did not subdivide a subquadrant if the OPC for its four children would require more bytes of computer storage than would the integer elevations of the grid cells within the patch. Minimum space-efficient levels for each order of OP are given in Table 1. Because our objective was to examine space-efficiency, our program simply counted the number of leaves of each level which are needed to represent the surface; in an actual application, the appropriate elements of the coefficients matrix **B** would be stored along with a key-number denoting the location of the quadrant, either in a pointer-based quadtree or a linear quadtree.

TERRAIN SAMPLES

We applied the methods discussed above to three topographic samples. Each sample was a 256 by 256 sub-sample of a U. S. Geological Survey 7 1/2 minute digital elevation model (Elassal and Caruso, 1983). The DEMs were collected as by-products of orthophoto quadrangle mapping, and all three have a grid cell size of 30 meters and a vertical height resolution of 1 meter. The two Pennsylvania samples were collected using a Gestalt Photo-Mapper II (or GPM-II; Swann and others, 1978; Elassal and Caruso, 1983); the sample from Oregon was produced using a semi-automatic B-8 stereo-plotter (Elassal and Caruso, 1983). Possible effects of data collection methods on DEM characteristics were discussed by O'Neill and Mark (1985).

The three test areas represent distinctly different types of topography, and have been used in previous DEM studies of fractals (Mark and Aronson, 1984) and topographic slope (O'Neill and Mark, 1985). The Blair's Mills (Pa) quadrangle is located in the Appalachian Mountains. Strong structural control has produced a series of aligned ridges and valleys oriented in a northeast-southwest direction. In contrast, the Keating Summit (Pa) quadrangle represents topography developed in the flat-lying sedimentary rocks of the Appalachian Plateau, showing little if any sign of structural control. Finally, the Adel (Ore) quadrangle is from the Basin-and-Range topographic province in south-eastern Oregon. In this area, the steep fault scarps and associated canyons contrast sharply with the gently-sloping plateau above and the flat valley floor below.

RESULTS

As noted above, the program we used began by breaking each 256 by 256 grid into 16 by 16 (level 4) subgrids. Then, for each of these, a surface of the current order was fitted to the data; if the RMSD was larger than the current threshold, the square was divided into four 8 by 8 squares, and the procedure was applied recursively. Table 2 presents the main results of the 54 runs of the program (3 DEMs times 3 RMSD thresholds times 6 polynomial orders).

As expected, for each order and for each DEM, the space required declines as the RMSD threshold increases. Unexpected was the apparent independence of space required and polynomial order (for fixed RMSD and DEM): each column in Table 2 contains values that are relatively constant. (We do, however, note that third-order polynomials were best in 4 cases, whereas first- and sixth-order polynomials were never the most space-efficient.) The fact that space-efficiency is almost independent of polynomial order suggests that, for a given RMSD threshold, the DEM has a fixed "information content", which can be expressed in bytes. The DEM can be approximated by many small patches containing simple surface polynomials, or by a smaller number of more complicated surfaces, and the two effects seem to cancel out. This should be the subject of further investigation.

The other pattern evident in Table 2 is that, for the 1-meter RMSD threshold, the Adel sample required far less space than the others. This might be in part due to the nature of the terrain, which (as noted above) consists of fault scarps, flat valley floors, and fault dip-slopes in the form of inclined planes. However, we believe that the difference is primarily due to the short-scale error characteristics of the data, which are chiefly dependent upon the method used to collect the data. The Gestalt

TABLE 2: SPACE REQUIREMENTS* FOR THE OP-QUADTREES FOR POLYNOMIAL ORDERS 1 THRU 6

Quadrangle:	Adel			Keating Summit			Blair's Mills		
RMSD:	1.0	3.5	7.0	1.0	3.5	7.0	1.0	3.5	7.0
1	57.7	24.6	11.3	96.3	37.6	16.3	96.6	38.3	13.2
2	53.3	20.3	10.0	90.8	28.8	12.9	93.9	36.7	11.3
3	53.6	19.7	9.5	99.3	28.0	9.6	98.4	30.8	11.5
4	48.0	19.8	12.4	94.5	24.5	12.3	97.4	32.8	12.5
5	50.0	21.3	16.8	88.5	20.2	16.8	88.7	33.2	16.8
6	60.3	24.1	22.3	92.2	23.6	22.3	94.7	31.9	22.3

* figures in table are required space as a percentage of space required by the original grids

Photo-Mapper (GPM-II) collects what is essentially a tree-top surface. The model contains error with a magnitude of about half the average tree height, and with a local structure similar to white noise. Such uncorrelated errors make it almost impossible for simple polynomials to approximate local areas when the maximum RMSD is set at 1 meter. The effect declines with increasing RMSD (see also Table 3, below). We expect that similar results would apply for other DEM data collected using the GPM-II.

In order to provide a more detailed evaluation of the interaction between maximum RMSD and space-efficiency, we fitted third-order polynomials to each of the three DEM samples using RMSD thresholds ranging from 1 to 10 meters; the results of these evaluations are presented in Table 3. The relation between space and RMSD for each DEM appears to be well-approximated by a power function (straight line on log-log graph paper). These curves could be compared to those generated for other DEM data structures, such as TINs prepared to approximate grids to within some pre-defined tolerance.

TABLE 3: SPACE REQUIREMENTS* FOR THIRD ORDER POLYNOMIALS

maximum RMSD (m)	Adel	Blair's Mills	Keating Summit
1	53.6%	98.4%	99.3%
2	31.1%	74.9%	54.1%
3	22.2%	39.6%	31.9%
4	16.9%	25.0%	2[illegible].4%
5	14.0%	17.9%	17.4%
6	11.2%	14.2%	12.1%
7	9.5%	11.5%	9.6%
8	9.1%	10.0%	8.7%
9	8.5%	9.1%	8.3%
10	8.2%	8.6%	8.2%

* figures in table are required space as a percentage of space required by the original grids

SUMMARY

For a fixed RMSD, the space requirements appear to be relatively independent of polynomial order. High-order polynomials fit large regions more easily, and thus the OP-quadtrees have fewer leaves; however, each leaf requires more space for the polynomial coefficients, and the two effects seem to cancel. If this can be shown to be true in general, the implication is that the order used does not matter. In that case, low-order polynomials should be used, since they can be computed more efficiently. When order is held constant, space requirements for the OP-quadtree appear to be an inverse power function of the RMSD criterion for low values of it (RMSD < 10 meters).

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