SCALE-BASED SIMULATION OF TOPOGRAPHY

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ABSTRACT

A model of natural terrain is proposed, which describes topography as the summation of scale-dependent periodic spatial structure, and scale-independent fractional noise. These two elements can be varied, and appear related to each other by the sampling theorem and the scale at which the model is applied. The scale-dependent component of the model can be calibrated for any given landscape using Fourier analysis, as can the scale-independent component by measurement of the fractal dimension of a specific piece of terrain. The model is invertible, allowing simulation of terrain with specific surface characteristics. This paper develops the model mathematically, gives a description of the advantages and disadvantages of the model, shows examples of the use of the model for terrain simulation using a test data set, and discusses the cartographic uses of the model in automated mapping systems.

INTRODUCTION

The representation of terrain has been subjected to cartographic research for many years, and this research has been intensified by the introduction of automated cartographic methods. Traditional methods, such as representation by contours, hypsometric tints, and hatchures (Imhof, 1982), have been supplemented by stereo-pairs, hill shading, anaglyphs, wire grid perspectives; and, more recently, natural perspective views (Schachter, 1980). Describing or analyzing terrain, however, has been left as the domain of the geomorphologist, except for a few cartographic 'rules of thumb'. Cartographers enforce terrain rules such as the law of V's, the avoidance of enclosed depressions, and the imposition of the fact that water tends to flow downhill.

It is the intention here to propose that the cartographer's domain of interest extends further than the mere measurement and representation of terrain, indeed extends as far as mathematically describing terrain in such a way that the description can be calibrated to characterize different types of natural topography, and can be inverted to simulate terrain.

There are direct benefits to cartography with this approach, such as the ability to synthesize particular types of terrain to test various representation methods or map interpretation skills, or to add synthetic detail to thematic maps (Morrison, 1986). In addition, the approach offers possible research potential in the areas of optimal sampling, optimal map scale selection, and a method to create 'initial condition' landscapes for further modeling and simulations. The production of simulated depictions of landscapes also has potential uses as a cartographic context for computer graphics displays such as flight simulators and games (Hamilton, 1986).

The mathematical model of terrain advanced here has its origins in work in geology and geography and the structural analysis of terrain, and borrows from two bodies of theory. The first of these is that of spectral analysis, in which the 'model' of terrain is geometric, and consists of sets of trigonometric functions. The second is the body of work following the application of the concepts of Benoit Mandelbrot in geology and cartography. In this work, the model of terrain is based on the concept of self-similarity, and incorporates the idea of the fractional dimension (Mandelbrot, 1977).

The link between these two bodies of theory lies in their treatment of scale. Scale has always been of primary concern in cartography, and cartographers are familiar with the problems of generalization and loss of resolution associated with changing map scale. Similarly, scale is of critical concern in remote sensing and photogrammetry, and is here directly related to the problems of automated feature detection and recognition. The proposed model treats scale as a two step problem. Some of the generalizations made in cartography at smaller scales are designed to eliminate from the level of cartographic detail elements of the landscape of a particular size. Examples are the generalization of inlets and the elimination of islands along coastlines. The implication here is that some aspects of the landscape, for the purposes of description I will call them 'forms', have a particular map scale at which they can be depicted as significant. The range of scales at which the forms can be depicted are given by the sampling theorem, since the map should be large enough to depict at least one whole 'form' at one extreme, and should have a resolution such that at least two observations or spatial measurements fall along the length of the form at the other extreme.

The scales at which 'forms' exist in the landscape can be quantified using Fourier analysis. Beyond the limits of the sampling theorem, we can say little about the form since it cannot be measured, except that forms close to but beyond these ranges will produce non-random variation in spatial measurements. In particular, the too-large forms will introduce 'regional trends' into the data, while too-small forms will give the impression of local roughness of terrain or terrain texture. A convenient way to model this type of variation is to use the concept of self-similarity. Self-similarity implies that the spatial structures inherent within a landscape repeat themselves at all scales, i.e. a part of the whole resembles the whole. Self-similarity has not been found in many actual landscapes, and in itself seems a poor model of terrain, at least on earth. This is because earth-forming and earth-moving processes have been at work on almost all landscapes, for varying amounts of time, and as a result the landscape bears the 'forms' or manifestations of the scales at which these processes operate or operated in the past. Where these processes are largely absent, such as on Mars, self-similarity seems to hold (Woronow, 1981).

To summarize, the implication of this reasoning is that topography contains spatial forms with distinctive scales or 'sizes'. If these forms could be measured and extracted from terrain, the resulting variation would be the combined effect of forms beyond the measurement ability of a particular terrain map. This residual variation should show no scale-related structure and would be self-similar.

Several measures exist and have been used to describe form in a general sense, among them the autocorrelation function (Cliff and Ord, 1969) and the variogram (Woodcock and Strahler, 1983). Really these functions are expressions of the *neighborhood property*, in that they reflect how objects are related to each other as a function of the spacing between them. The proposed model of terrain uses Fourier analysis, and therefore suggests a trigonometric-periodic nature to the variation in elevation over space. The model could be described as being *scale-dependent*, since the separation of characteristic 'forms' by scale is the basis for the model.

When no scale-dependence exists, the neighborhood property is similar at all scales, implying that the phenomenon is self-similar, fractal, or *scale-independent*. Scale-independent phenomena are identical in structure (or lack of it) at all measurable scales but are not totally random, since they reflect structures at scales beyond those measurable. Since we can only deal with scales at which we can make measurements, we can usually only prove statistical self-similarity or scale-independence over a stated scale range.

Spatial form in a natural landscape is rarely distinct. Several forms at a variety of scales can exist together. Measurement of these forms must involve separation of the forms by scale and location. If these forms are extracted from a particular piece of terrain the residual could be considered purely random variation. Sources of this variation are measurement error, and variation due to stochastic processes; but, most significantly, variation exists because of scales beyond the limits set by the sampling theorem. The residual variance in terrain in this case is random at the scales over which it can be measured, so whatever scale is used for measurement the form will appear the same. Such a landscape has been termed self-similar (Mandelbrot, 1977) and can be simulated by fractional Brownian motion. It is not likely that fractal characteristics are displayed by terrain as a whole, a fact shown by Mark and Aronson (1984). If terrain itself was self-similar, the whole surface could be modeled by this single process (Goodchild, 1982).

Measurements describing this fractal type of form depend on the size of the measurement instrument, such that the relationship between measurements and the instrument size form a predictable ratio which can be used to determine the terrain's fractional dimension. The fractional, fractal or Hausdorff-Besicovich dimension is a real number, unlike the integers associated with dimensions in Cartesian geometry, and varies between 2 and 3 for a surface.

The natural landscape is a complex amalgam of spatial forms, each with its own associated scale, plus the scale-independent variation with fractal characteristics. It can be thought of as consisting of two elements, the scale-dependent part, reflecting the structures of objects at the various scales discernible in the landscape, and the fractal part, which is scale-independent. Measurements of spatial form therefore must be able to separate these elements, to split the scale-dependent part into its various scale objects and to quantify the fractal dimension of the residual variation. Simulation of terrain should involve inverting these measurements to produce a synthetic piece of terrain which is consistent with the derived measurements.

METHODOLOGY: MEASUREMENT AND MODELING OF SCALE-DEPENDENCE

Cartographic symbolization of three dimensional objects is based on the topographic surface, and its discrete representation, the Digital Elevation Model (DEM). The Geological Survey publishes these data, at scales of 1:250,000 and 1:24,000, with spatial resolutions of three arc seconds and 30 meters respectively. These data are integer elevations in meters, sampled at the nodes of a regular square grid. The DEM can be thought of as cartographic measurement of terrain at a particular resolution (the grid spacing), at a particular map size (the number of rows and columns in the DEM). DEMs have the properties of elevation, slope, volume, surface area, direction and angle of dip, texture, pattern and roughness; and they show the effects of scale-dependence and independence. Since topographic surfaces are assumed to be continuous, topology will be ignored, and all values on the surface are unique (no underground caves or overhangs).

Scale-dependencies are measured for calibration of the model using Fourier analysis of a DEM. This method is based on the concept that functions defining the surface of a DEM can be abstracted into sets of trigonometric series. Any surface, regardless of complexity, can be modeled by the sum of a set of sine and cosine waves with different wavelengths and amplitudes. This is indeed true if we can use an infinite set of trigonometric functions, but in reality we are constrained by the sampling theorem to wavelengths of a given range, the range from the fundamental to the Nyquist frequency. Most simple functions are well described by the sum of only a few sets of sine and cosine functions, and even some very complex functions can be adequately described using as few as ten sets (Tobler, 1975).

Fourier analysis (Rayner, 1971) computes a full set of trigonometric functions for the given range of wavelengths, and determines each one's contribution to the description of the curve under analysis. Those that make large contributions are valuable and can be used to invert the measurement process. Associated with each of these 'significant' trigonometric functions is a spatial 'wavelength'. This is the scale at which a periodicity exists. Fourier analysis can be used to reveal which scales are present in the data, and is useful in calibrating a scale-based model of terrain. Examples of the use of Fourier analysis to do this are provided by Davis (1973), Rayner (1972), and Bassett (1972).

Fourier analysis is directly invertible. This means that given the descriptors of the trigonometric components of the curve, the curve can be reconstructed with a level of accuracy dependent upon the number of descriptors used. If only the most significant descriptors are used in the reconstruction, the effect is to rebuild a regular spatial form, from which has been removed both variation beyond the ranges of the sampling theorem and that residual variation not attributable to the principal harmonics.

The same technique of Fourier analysis can be applied in one and two dimensions. Early work in Geography and Cartography on this problem was performed by Moellering and Rayner (1979), exclusively in one dimension, or as two one-dimensional series. Alternately, use of a two dimensional version of the functions allows the analysis of surfaces (Clarke, 1984; Rayner, 1972). The equation for a one dimensional series is:

$$z_i = \sum_{k=1}^{k=km} (A \sin k \, \omega \, x_i + B \cos k \, \omega \, x_i) \tag{i}$$

where spatial resolution x is given by:

$$x = L \;/\; k \tag{ii}$$

and L is the "length" of the map. The value omega is termed the angular frequency, or in spatial terms the number of cycles that a given periodicity goes through in a given distance, and is given by:

$$\omega = \frac{2\,\pi}{L} \tag{iii}$$

The value km is the limit of the Fourier series, and is reached when x approaches twice the resolution of the data. The more complex two dimensional series is given by:

$$z_{x,y} = \sum_{k=1}^{k=km} \sum_{j=1}^{j=jm} (A_{kj} C_k C_j + B_{kj} C_k S_j + C_{kj} S_k C_j + D_{kj} S_k S_j) \tag{iv}$$

where the sine and cosine terms are given by:

$$C_k = \frac{\cos(\,2\,k\;\pi\,x\,)}{L_x} \tag{v.1}$$

$$C_j = \frac{\cos(\,2\,j\;\pi\,y\,)}{L_y} \tag{v.2}$$

$$S_k = \frac{\sin(\,2\,k\;\pi\,x\,)}{L_x} \tag{v.3}$$

$$S_j = \frac{\sin(\,2\,j\;\pi\,y\,)}{L_y} \tag{v.4}$$

and where L is the length of the series in x and y as given by the subscripts. The power spectrum for the two dimensional series is an array, P given by:

$$P_{k,j} = A^2_{k,j} + B^2_{k,j} + C^2_{k,j} + D^2_{k,j} \tag{vi}$$

The decision has to be made as to what constitutes a 'significant' harmonic to be used in the reconstruction and considered a scale at which a form occurs. Peaks in the power spectrum are used to signal a significant harmonic. Generally, if terrain is reconstructed from only a few harmonics, the surface will be both smooth and simple. The more harmonics are included in the inversion, the more the inverted Fourier surface will resemble its original. A quantitative basis for the inclusion of harmonics is the proportion of each harmonic's contribution to the total power, a value similar to a proportional contribution to variance.

Fourier analysis is comparatively simple to perform, shows significant scales directly, is applicable to two and three dimensional data, and is invertible, allowing a 'form' surface to be rebuilt from the significant parts of the trigonometric series. Two computational forms of the Fourier transform are in current use. The discrete transform has the advantage of conceptual simplicity and applicability to all data ranges. The fast-Fourier transform (Bloomfield, 1976), however, is vastly superior to the discrete transform in computational efficiency, but suffers from the problem of being applicable to only specific dataset sizes.

METHODOLOGY: MEASUREMENT AND MODELING OF SCALE-INDEPENDENCE

Measurement of scale-independence implies measurement of the fractal dimension. Fournier et al. (1982) provided the necessary algorithms for simulating surfaces from the fractal dimension for surfaces, and Dutton (1981) used an algorithm for lines and polygons.

This implies that the inverse transform is feasible, if the fractal dimension has been empirically derived beforehand.

Several different but related methods exist for the computation of fractal dimensions. The first set of methods relate to the computation of the fractal dimension of lines (and polygons). The simplest method is the 'walking divider' method. This method simulates the analog process of opening a set of dividers to a spacing d, then walking the dividers along the line, taking a total of N steps. Actually, N should contain a fraction of a step at the end of the line to reach the actual end point. The measured length of the line (L) is then N times the distance d. If the process is repeated, each time doubling the spacing of the dividers d, a set of pairs of observations of L and d can be derived. The natural logarithm of each of these values is taken, and a least squares linear regression is used to estimate the slope of the linear relationship between them. This slope (b), is then used to compute the fractal dimension f, which is simply 1 - b. Since the value of b is negative (i.e. the length of the line decreases as the spacing of the dividers increases) f is greater than 1 and is constrained to be less than 2.

A computer implementation of this algorithm has been performed by the author, and other versions exist (Shelberg et al., 1982). Goodchild (1982) has used a cell counting method, in which the number of joins between cells above and below a certain elevation is used as a proxy for line length. The same author also used a simple shape measure (perimeter length divided by area) for the areas within contours to obtain similar fractal dimensions to the walking dividers method.

Computation of fractal dimensions for surfaces has been performed in a variety of ways. The simplest method is to take advantage of the fact that any horizontal cross-section of a fractal surface should form a polygon with a fractal dimension which is the fractal dimension of the surface minus one. Since contours provide convenient cross-sectional polygons, these have been most frequently used (Goodchild, 1982; Shelberg and Moellering, 1983), while Burrough (1981) has used vertical cross-sections. Polygons within contours at sea level reflect coastal landforms, while those at high elevations reflect fluvial, mass wasting and glacial landforms. Scale-independence, statistical or otherwise, seems unlikely over these ranges. Shelberg and Moellering (1983) attempted to rectify this problem by interactive selection of the chosen isoline.

Mark and Aronson (1984) used the variogram as a mechanism for the computation of the fractal dimension. To compute the variogram for a surface, Mark and Aronson used sets of randomly-located points within the largest circle contained within the map area. About 32,000 such points were used for each surface. From the elevations of these points, the variance was computed as a function of distance. This relationship was used in a log-log regression to estimate the slope (beta) and thus the fractal dimension f as:

$$f = 3 - (\beta / 2) \qquad \text{[vii]}$$

Mark and Aronson computed fractal dimensions for seventeen digital elevations models with 30 meter resolution for a variety of landscapes. Of these, only one surface showed self-similarity at all scales measured, this being the Shadow Mountain 7 1/2 minute quadrangle in the Colorado Rocky Mountains. It may be possible that computations on this DEM involved data errors (Mark, personal comm., 12-14-86). All others to some extent, and some very strongly, showed periodicities due to spatial forms at particular scales. These results strongly suggest the validity of the proposed model.

A similar method was suggested by Burrough (1981). Burrough used the variogram, the Fourier power spectrum, the covariance, and the variance to compute fractal dimensions of vertical cross sections for a variety of different environmental phenomena. The use of the Fourier power spectrum was suggested by Fournier et al. (1982) as one of three possible approaches to fractal modeling and was used by R. Voss for the illustrations in Mandelbrot (1977). The method has been extended for use in image segmentation for image processing by Pentland (1984). The actual method consists of first computing the Fourier power spectrum, and then taking the slope (beta) of the log-log relationship between the power series and distance. In this case, the fractal dimension is given by:

$$f = 3 - (\beta / 2) \qquad \text{[viii]}$$

The similarities between this and the variogram approach are more than coincidence, since the variogram is the algebraic equivalent of the power spectrum. Burrough made estimates of fractal dimensions from observed data on variograms, block variances, covariance and the power spectrum, and found many different environmental data to show scale-independence. Burrough concluded "although some environmental data do appear to display the fractal property of statistical self-similarity at all scales, there are also many that show self-similarity over a limited range of scales, or over a few widely separated scales".

Fourier analysis allows the computation of the two-dimensional power spectrum for terrain. At every point in the power spectrum array, the one-dimensional distance of the harmonic combination can be computed, since for harmonic pair kx in the x direction and ky in the y direction:

$$d_{kx,ky}=[(L_x \;/\; kx)^2 + (L_y \;/\; ky)^2]^{(1/2)} \qquad \text{[ix]}$$

The fractal dimension can then be computed from a log-log fit of power and distance, taken across all values of k from 1 to the maximum, which is given by twice the resolution of the data.

$$\frac{d\ (\log P)}{d\ (\log d\ )} = \beta \qquad \text{[x]}$$

Beta can then be substituted into equation viii.

Another method for computing the fractal dimension of surfaces has been developed by the author (Clarke, 1986), and has been called the Triangular Prism Surface Area method. This technique is a three dimensional equivalent of the walking dividers method, and uses the surface area at a range of resolutions for a DEM to estimate the fractal dimension. In this case, the log-log relationship used to estimate the fractal dimension is that between the total area of the DEM surface, and the size of the cells into which it can be divided equally by powers of two.

The attractions of the Fourier power spectrum for the computation of the fractal dimension within the context of this work are multiple. First, the power spectrum is computed for a surface as part of the means of extracting significant periodicities in the scale-dependent part of the analysis. Secondly, this measure may be the best at estimating the fractal dimension since a larger number of scales are used in the log-log regression. Thirdly, and most importantly, the Fourier coefficients can be made fractal, and then the transform can be inverted to provide a simulated surface with fractal characteristics.

THE MODEL

The model of terrain postulated here may be expressed as follows. Elevation (z) within a DEM is given by:

$$z_{x,y} = T_{x,y} + cF_{x,y} + H_{x,y} \qquad \text{[xi]}$$

where:

$$T_{x,y}=\beta_0+\beta_1 x+\beta_2 y \qquad \text{[xii]}$$

defines a linear trend (very low frequency harmonic), and

$$F_{x,y}= \sum_{k=1}^{k=km} \sum_{j=1}^{j=jm} (A_{kj}C_k C_j+B_{kj}C_k S_j+C_{kj}S_k C_j+D_{kj}S_k S_j) \qquad \text{[xiii]}$$

The Fourier coefficients are provided in one of two ways. For significant harmonic pairs, the coefficients are derived empirically from real DEMs by Fourier analysis. During the inversion, insignificant harmonics are given zero Fourier coefficients, resulting in an inverse transform DEM showing scale-dependency only, H. In the case of scale-independency, the Fourier coefficients are given by:

$$A = B = C = D = \sqrt{[P_{k,j}/4r^2]} \qquad \text{[xiv]}$$

where r is 1 if k and j are zero, 2 if one but not both are zero, and 4 otherwise. The power is given by:

$$\log P_{k,j} = \alpha + \beta \log d_{k,j} \quad [xv]$$

The values of A, B, C, and D in equation xiii can be either positive or negative, and if desired could be assigned random values such that their sum fell within limits. In the implementation of the model here, they were assigned values which were identical to those in the empirical Fourier analysis.

Finally, the relative contribution of the scale-independent component was controlled by scaling the range by a constant c. This was done because the linear regression provided a new set of Fourier coefficients for the full range of harmonics, including significant harmonics and those with very high frequencies. Significant harmonics often fall among the longer wavelengths, and can significantly influence the trend of the regression. Rather than filtering the data after the inversion to remove the high frequencies, instead the harmonic component was scaled by a factor designed to make the new surface mean and variance similar to the original. Thus the inverse transform involved two separate inverse Fourier transforms for the scale-dependent and the scale-independent components.

TEST APPLICATION

The model described above was tested using data from the Bell and Volfe Canyon areas in California's San Gabriel Mountains. This dataset had a fractal dimension of 2.193 computed by the Triangular Prism Surface Area method. Nine significant harmonics were found by the Fourier analysis for the area to have percentage contributions to the variance over 2.5 percent, with spatial wavelengths of 4 025 meters, 3 600 meters, 2 546 meters, 1 801 meters, and about 85 meters. These harmonics collectively contained 59.7% of the surface variance. Six of these harmonics made percentage contributions to the variance of over five percent.

A weak linear trend (r-squared of .38) was subtracted from the data which dipped to north north east. The remainder of the surface was subjected to Fourier analysis, and a log-log regression was performed between the raw power spectrum and spatial distance for each harmonic pair. The result was a weak linear relationship (r-squared of .32) with a slope of 1.162. Using equation viii, this yields a fractal dimension for the scale-independent part of the surface of 2.42, higher than the previous estimate. This regression was used to generate estimates of the Fourier coefficients in all harmonics, and the Fourier transform was inverted, scaled to reduce the variance to the same range as the original data, and added back to the linear trend. Finally, the inverted Fourier transform using the significant harmonics alone was added. Figure 1 shows a hill-shaded depiction of the Bell/Volfe Canyon data. Figure 2 shows the inverse Fourier transform using only the harmonics containing more that 2.5 percent of the surface variance, also hill-shaded. Figures 3 and 4 show the full application of the model. In the case of figure 3, a cutoff of 2.5 percent was used as a criterion for the inclusion of a harmonic as significant. In figure 4, a value of 5 percent was used.

DISCUSSION

The model proposed has various advantages and disadvantages for use as a terrain simulator. Advantages include the use of a single method, Fourier analysis, to incorporate two separate scale-dependent and scale-independent elements; the ease of inversion of the Fourier transform, the ability to base the model on parameters which can be measured from actual terrain, and finally, the fact that while a limited scale range is used here, the scale-independent component of the model makes the extension of the model across scales possible. The latter is important in computer graphics applications, where simulations are frequently required to zoom in and out on an area.

Disadvantages include the inefficiency of the current algorithm, which uses a discrete Fourier transform; the weak fit of the log-linear regression, which suggests that the residual may not be purely fractal; and the need to scale the inverted fractal surface back to the original surface variance range. Each of these shortcomings will be approached in future work.

In many respects, a terrain model can be judged by whether it produces terrain which 'looks right' in spite of the theoretical model from which the terrain was simulated. Nevertheless, the calibration of the model for different types of terrain in different parts of the world should produce statistics of geomorphological as well as cartographic interest.

The simulation of terrain may have applications in future automated mapping systems. Where thematic information is being depicted, for example, the precise type of terrain shown as background information need not be strictly accurate, especially at small scales. In these cases, an automated cartographic system could call up synthetic terrain with specific characteristics, rolling hills for example, and display them as relative topographic symbols. Alternatively, the system could provide synthetic landscapes for the development of context-free symbolization, which could then be transferred to a particular landscape or used to test the effectiveness of different representational methods. A distinct advantage of the scale-based model is that such systems could use the same model, regardless of the scale at which a particular map was to be produced.

Figure 1: Hillshaded Image of Bell Canyon DEM (120 x 120 pixels)

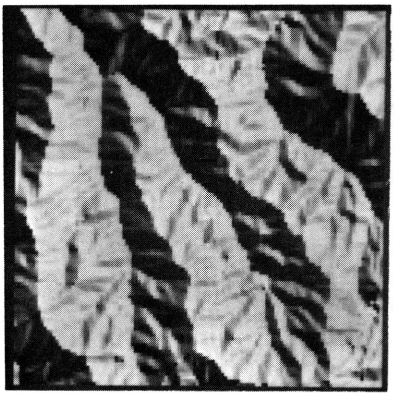


Figure 2: Bell Canyon Reduced to the Significant Harmonics

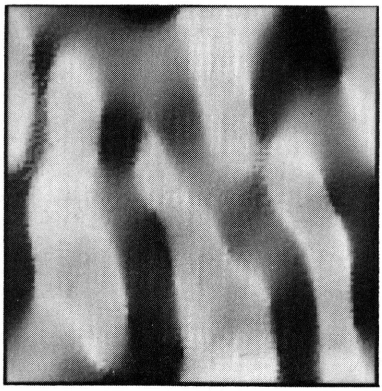


Figure 3: Simulated Terrain, Using Nine Harmonics, Fractal Dimension 2.42

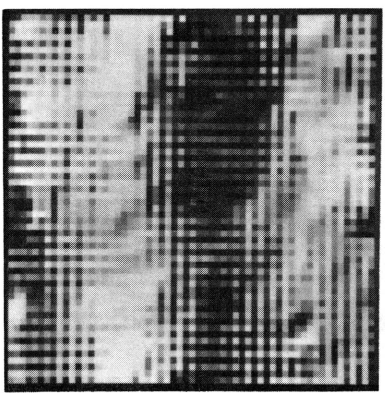
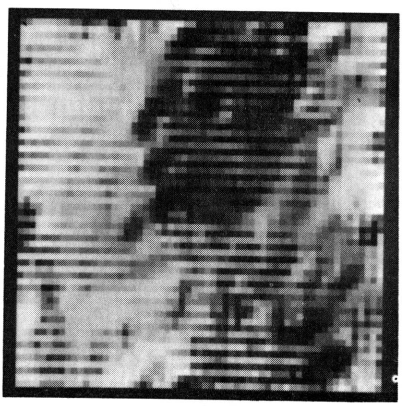


Figure 4: Simulated Terrain, Using Six Harmonics, Fractal Dimension 2.42



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