INCORPORATING THE LABORDE PROJECTION INTO AN EXISTING CARTOGRAPHIC SOFTWARE PACKAGE

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ABSTRACT

A modular method for incorporating the Laborde projection into an existing cartographic software package, which requires the minimum amount of extra coding, is described. The method exploits the natural modularity in the definition of the Laborde conformal projection and utilizes the existing transformations likely to be found in any standard software library of map projections. The few missing formulas that would be necessary to complete the Laborde mapping equations (but are hard to find in the literature - such as the scale factor equation) are provided. Also, in case the suitable conformal latitude routines are not available, an alternative approach (using Mercator equations) to the transformation from ellipsoid to sphere is proposed.

INTRODUCTION

Cartographic projections enable the representation of the curved surface of the ellipsoidal (or spherical) Earth onto the flat surface of the map. Some projections are used more often than others, mainly based on their usefulness, but also based on the traditionally established standards. Lambert Conic Conformal and Transverse Mercator projections became standards for large scale mapping. These projections are always easy to find in any standard cartographic software package.

Some other projections are almost extinct from today's cartographic practice, often because they were not useful, but sometimes because they never gained enough attention. The Laborde conformal projection, used for Madagascar Grid, is a good example of a projection with the single implementation: for Madagascar only. This projection is not likely to be found in a standard cartographic software package, partly because the computer ready projection equations are not likely to be found in any modern cartographic textbook! Yet the projection has all the desired geometric properties of the Oblique Mercator projection, without being so undesirably sensitive to the small changes in the defining azimuth of the central line, as is the well known Hotine's version of the Oblique Mercator projection (compare Hotine 1947). In many respects, the Laborde approach produces equations that are more numerically stable than the usual Oblique Mercator's equations.

The cartographer who wants to include the Laborde projection into his cartographic software package has to cope with the old-fashioned, often tabular descriptions (mainly in French and German) of the Gauss-Schreiber projection, a projection similar but not identical to the Transverse Mercator projection.

This paper describes the alternative approach, in which the Laborde projection equations are constructed from the separate functional modules, likely available in any standard cartographic software library. This approach utilizes the intrinsic modularity in the original definition of the projection, and minimizes the amount of new computer code required.

DEFINITION OF LABORDE PROJECTION

The Laborde conformal projection was formulated by Commander J. Laborde (1928) as a triple projection, designed for Madagascar, an island elongated in a direction which is at an angle to the meridian and to the parallel. The properties of the projection are controlled by a set of projection parameters, in a manner similar to the well known Oblique Mercator projection (Hotine 1947):

lat0 - the latitude of origin
(-18º54' for Madagascar)

lon0 - the longitude of origin
(46º26'13.95" for Madagascar)

Az - the azimuth of the axis of strength
(18º54' for Madagascar)

k_0 - the scale reduction factor at origin
(0.9995 for Madagascar)

FE - False Easting
(400000m for Madagascar)

FN - False Northing
(800000m for Madagascar)

Despite the fact that both, Laborde's and Hotine's projections have a similar set of defining parameters, conceptually they have been constructed in a different manner. The Laborde version uses the conformal sphere as an intermediate surface, whereas the Hotine's version uses a special geometric form called aposphere as an intermediate surface.

Computationally, the obliquity of the Laborde projection is controlled by the azimuth at the projection origin, normally at the center of the map, whereas the Hotine's equations indirectly use the azimuth at the equator of aposphere, positioned usually thousands of miles away from the mapped area. As a result, the Laborde equations are numerically more stable with respect to the small variation in the defining azimuth than the Hotine's equations are. Also, Laborde's equations are well defined for azimuth angles close or equal to 0º or 90º, whereas Hotine's equations are not.

As a triple projection, the Laborde equations can be decomposed into three separate conformal projections:
1. ellipsoid to sphere, according to Gauss representation of the second kind (Gauss 1844)
2. sphere to plane, using the spherical Transverse Mercator equations
3. plane to plane, according to the Laborde complex third degree polynomial (Laborde 1928).

The first step defines the intermediate surface of the conformal sphere. The radius of the sphere is the Gaussian Mean Radius of Curvature calculated at the projection origin. The latitude of the projection origin is maintained true to scale (standard parallel), and the scale differs very little from unity in a wide zone surrounding the standard parallel (by design the scale error is maintained close to zero as the quantity of the third order with respect to the angular distance from the standard parallel).

Although the second step alone, the Transverse Mercator projection, does not require any explanation, it is an interesting fact that the first step and second together produce the Gauss-Schreiber projection of the ellipsoid to the plane, which differs slightly from the ellipsoidal Transverse Mercator projection, in that the central meridian is not quite true to scale.

The third step is the conformal transformation from the intermediate plane of the Gauss-Schreiber projection to the final plane of the Laborde projection, designed to reduce the scale error along the chosen oblique axis at the expense of losing an "almost true to scale" meridian generated by the Gauss_Schreiber projection. This is achieved through the (complex) polynomial transformation of the plane, rather than through an ordinary planar rotation.

IMPLEMENTATION STEPS

In this section the software implementation steps for the Laborde projection will be outlined. Only the equations not likely to be found in a standard cartographic software library will be given. These few equations would have to be coded in the form of subroutines and added to an existing software library. The final code for the Laborde projection should then be composed of successive calls to the existing routines, precisely in the order implied by the original definition of the projection.

Step 1. Ellipsoid to sphere

The implementation of this step depends on the availability of the appropriate conformal latitude and longitude subroutine. There are (infinitely) many ways to conformally project ellipsoid to sphere. The Laborde projection specifically requires the application of the Gauss equations of the second kind (Gauss 1844). However, the most commonly known conformal latitude equations are those used in Adams (1921, p. 18,84), discovered by Lagrange in 1779. The Lagrange representation differs from the Gauss representation of the second kind in that it produces bigger scale errors as the angular distance from the standard parallel increases. Therefore, the Lagrange representation cannot be used for the Laborde projection.

Concluding this step, if the conformal latitude and longitude subroutine which uses precisely the Gauss representation of the second kind is available - it should be used to transform the ellipsoid to the conformal sphere.

Otherwise, the computations in this step may be accomplished, in three separate stages, by the following procedure (which follows directly from the Gauss original definition, and algorithmically utilizes the ordinary Mercator projection equations):

1. The conformal transformation of latitude and longitude (lat,lon) on the ellipsoid to the isometric plane (x,y) may be accomplished by using the forward equations of the ellipsoidal Mercator projection. The parameters to the Mercator subroutine should specify the eccentricity e, the unit equatorial radius a=1, the (Mercator) origin (lat=0,lon=lon0), and the equator true to scale.

In the formulas below, the latitude and longitude coordinates on the ellipsoid are denoted by (lat,lon), the respective latitude and longitude coordinates on the conformal sphere are (LAT,LON), the ultimate origin point of the Laborde projection on the ellipsoid is at (lat0,lon0), and the respective origin on the conformal sphere is (LAT0,LON0).

2. The conformal transformation of the (ellipsoidal) isometric plane (x,y) to the (spherical) isometric plane (X,Y) is accomplished by the Gauss linear equations (Gauss 1844):

$$X = c * x$$
$$Y = c * (y + dy) \tag{1}$$

where the scale and shift parameters should be precalculated as the projection constants:

$$c = [1 + (e^2 \cos^4(\mathrm{lat0})) / (1-e^2)]^{1/2} \tag{2}$$

$$dy = Y0 / c - y0 \quad , \tag{3}$$

where the isometric latitudes y0 and Y0 in equation (3) may again be evaluated using Mercator projection equations.

The ellipsoidal isometric latitude y0 is computed as the Northing value obtained by applying the forward equations of the ellipsoidal Mercator projection to the Laborde origin (lat0,lon0). The parameters to the Mercator subroutine should specify the eccentricity e, the unit equatorial radius a=1, the (Mercator) origin (lat=0,lon=lon0), and the equator true to scale.

The spherical isometric latitude Y0, needed in equation (3), is computed as the Northing value obtained by applying the forward equations of the spherical Mercator projection to the origin point on sphere (LAT0,LON0), where, from Gauss conditions, LAT0 should be computed as

$$LAT0 = \arcsin(\sin(lat0)/c) , \tag{4}$$

and LON0 = 0. The parameters to the Mercator subroutine should specify the eccentricity e=0 (for sphere), the unit radius a=1, the (Mercator) origin (LAT=0,LON=0), and the equator true to scale.

3. The conformal transformation of the (spherical) isometric plane (X,Y) to the resultant latitude and longitude (LAT,LON) on the conformal sphere may be accomplished by using the inverse equations of the spherical Mercator projection. The parameters to the Mercator subroutine should specify the eccentricity e=0 (for sphere), the unit radius a=1, the (Mercator) origin (LAT=0,LON=0), and the equator true to scale.

After the above steps, the resultant (LAT,LON) coordinates refer to the conformal sphere, precisely as implied by the Gauss representation of the second kind (Gauss 1844).

Step 2. Sphere to Gauss-Schreiber plane

For this step the spherical Transverse Mercator equations are appropriate. The parameters to the Transverse Mercator subroutine should specify the radius R which is equal to the Gauss mean radius R_o associated with the conformal sphere, and evaluated at the latitude of origin, lat0:

$$R_o = a\,(1 - e^2)^{1/2} \,/\, (\, 1 - e^2 \sin^2(lat0)\,) \, . \tag{5}$$

Other parameters should specify the eccentricity e=0 (for spherical equations), the (Transverse Mercator) origin LAT=LAT0 (given by equation (4)), LON=0, and the scale reduction factor at the origin k_o=1.

The resultant coordinates on this intermediate plane are precisely the Gauss-Schreiber coordinates of a (double) projection of the ellipsoid on the plane, similar (but not identical) to the ellipsoidal Transverse Mercator projection.

Step 3. Gauss-Schreiber plane to Laborde plane

This step should be programmed in the form of a subroutine implementing Laborde's conformal polynomial equations. These equations will be given here in the order of calculations.

Given the azimuth Az of the axis of strength (equivalent to the central line in the Oblique Mercator projection), evaluate the projection constants A and B:

$$A = (1 - \cos(2\ Az)) \ / \ (12\ R_o^2) \tag{6}$$
$$B = \sin(2\ Az) \ / \ (12\ R_o^2). \tag{7}$$

Then, for any given Gauss-Schreiber coordinates (x,y), the mapping equations, which produce the Laborde coordinates (X,Y), are

$$X = x + A\ f1 + B\ f2 \tag{8}$$
$$Y = y - B\ f1 + A\ f2 \tag{9}$$

where

$$f1 = - x^3 + 3xy^2 \tag{10}$$
$$f2 = - 3x^2y + y^3 \ . \tag{11}$$

Of course, as in any mapping equations, the final X,Y coordinates may be (uniformly) scaled down by the central scale reduction factor k_o (k_o=0.9995 for Madagascar), and the appropriate False Easting, False Northing may be added for the positive coordinates range.

Improving numerical stability

The large numbers that could be possibly generated in equations (10) and (11) may be easily avoided by the following modifications:

a) in Step 2, the call to the spherical Transverse Mercator equations should specify the radius paramotor R equal to 1 instead to R_o of equation (5),

b) in Step 3, the Laborde constants A and B (equations (6)(7)) should be evaluated using R_o=1, and the resultant Laborde coordinates of equations (8)(9) should be post-multiplied by the actual R_o, as properly determined in (5).

NOTE ON INVERSE EQUATIONS AND SCALE FACTOR COMPUTATION

The inverse mapping equations for the Laborde projection should be implemented by using the respective inverse equations for steps 3, 2, and 1 of the forward procedure. Again the assumption is that the inverse equations of the (spherical) Transverse Mercator projection and the Mercator inverse projection equations are available, and should be used in steps 2 and 1 (whenever applicable). The remaining steps require some additional explanation.

Beginning the inverse process with Step 3, the inverse of the Laborde conformal polynomial equations (8)(9) is accomplished by numerically solving for the unknown Gauss-Schreiber coordinates (x,y), using the given

Laborde coordinates (X,Y) as constants, from the system of nonlinear equations (8)(9), by using the method of simple iteration, also known as the method of fixed-point iteration (Burden, at al 1981). The initial approximation $(x_k, y_k)_{k=0} = (X,Y)$ is appropriate, where (X,Y) denotes the initial Laborde's Easting, Northing coordinates, from which the False Easting and False Northing, the scale factor k_0, and the Gaussian radius R_0 (equation (5)) were removed. In the case of Madagascar Grid (Laborde Projection Tables 1944), only two iterations are necessary to achieve the required accuracies . However, in the context of this paper, in the general application of the Laborde projection, it is better to allow for as many iterations as necessary for the complete numerical convergence.

In the final step of the inverse Laborde equation (conformal sphere to ellipsoid, the inverse of Step 1), if the Gaussian conformal latitude equations are not available, the ordinary Mercator equations are used again in a precisely inverse order to that described in the forward equations. In this case, the inverse form of the linear equations (1) must be used.

The equations for the scale factor k as a function of lat, lon on the ellipsoid are derived from the fact that a sequence of conformal transformations, performed in succession, produces a conformal transformation with the resultant scale factor equal to the product of the individual scale factors. Again the scale factor equations of the Transverse Mercator projection and the regular Mercator projection (if applicable) are obtainable from any standard software package. The scale factor associated with equation (1) is of course the Gauss constant ratio c given by equation (2).
The derivation of the scale factor associated with the Laborde conformal polynomial (8)(9) is only a little more complicated. Using the complex numbers notation, mapping (8)(9) may be written as

$$Z = z + (B+Ai)z^3 \qquad (12)$$

where

$$Z = Y + Xi, \quad z = y + xi, \quad i^2=-1 \qquad (13)$$

From the theory of conformal mapping (analytic functions) we have a direct expression for the scale factor at the arbitrary Gauss-Schreiber coordinates (x,y):

$$k(x,y) = [dZ/dz]_{(x,y)} = 1 + 3(B+Ai)(y+xi)^2 \qquad (14)$$

This equation may be programmed using complex arithmetic or treating separately the real and imaginary parts.

Remark: in programming of the scale factor sequence, it is important to always transform the point of evaluation, given at a start as an arbitrary point (lat,lon) on the ellipsoid, to the intermediate surface appropriate for the transformation component being evaluated.

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