THE COMBINATORIAL COMPLEXITY OF POLYGON OVERLAY

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ABSTRACT

The number of elementary connected regions arising from polygon overlay of two or more map layers is an important value to have in planning for data storage and in making processing time estimates for overlay applications. That number may be computed directly from the line graphs of the two (or more) layers and from the intersection graph(s) of those line graphs. A formula for that computation is derived using tools of algebraic and combinatorial topology which relate the connectivity of a union of sets to the connectivity of the sets themselves and their intersection. The result and the formula may be stated as follows:

Suppose X is the line graph (1-skeleton) of a map. Regard X as embedded in the plane. Let $r(X)$ be the number of regions of the plane separated by X. Then $r(X)$ is the number of connected components in the planar complement of X; $r(X)$ is also one more than the maximum number of independent cycles in the graph X; and $r(X)$ is easily computed using standard graph traversal techniques for counting independent cycles. Let $c(X)$ be the number of connected components of X.

If A and B are the line graphs of maps to be overlaid, then $A \cup B$ is the line graph of the overlay; and:

$$r(A \cup B) = r(A) - c(A) + r(B) - c(B) - r(A \cap B) + c(A \cap B) + c(A \cup B)$$

All of the values on the right hand side of the equation can be readily computed using standard graph traversal and line intersection algorithms to obtain the desired value, $r(A \cup B)$, the number of regions after overlaying.

1. INTRODUCTION

The fundamental naive combinatorial question regarding polygon overlay is the following: If I overlay a map of n regions on another map of m regions, how many regions are there in the composite map? The possible answers are: any number that is not smaller than $\max\{m, n\}$. Hence, the answer that we give cannot be a number or even a bound. We relate the number, instead, by an exact formula, to the number and kind of line intersections that occur. In so doing, we transform the problem into one that is more amenable to analysis and to establishing constraints. In this paper we present some methods and results of algebraic topology that illustrate the nature and the methods of dimensional duality for addressing some of the global questions in mathematical cartography. We do not pretend to develop theory of algebraic topology in any detail here–indeed, to arrive at our small result, we must skim over a great deal of mathematics. The interested reader is directed to Henle [1] for more of the topological and combinatorial details and to Hu [2] for a more complete exposition of algebraic concepts.

This paper introduces and describes a limited number of tools of algebraic topology–a sufficient number to derive the formula that relates intersections to the number of regions of the overlay.

2. PRELIMINARIES: CHANGING TOPOLOGY TO ALGEBRA

2.1. Basic Concepts in Algebraic Topology

Algebraic topology is the area of mathematics that examines algebraic properties of algebraic objects derived from topological spaces. Spaces which are topologically equivalent have the same collection of algebraic objects associated with them; and mappings between topological spaces have associated with them mappings between the corresponding algebraic objects. Topological problems are converted to algebraic problems under the described association (formally this association is called the functor from the category of topological spaces and continuous functions to the category of groups and group homomorphisms or the category of rings and ring homomorphisms or some other algebraic category).

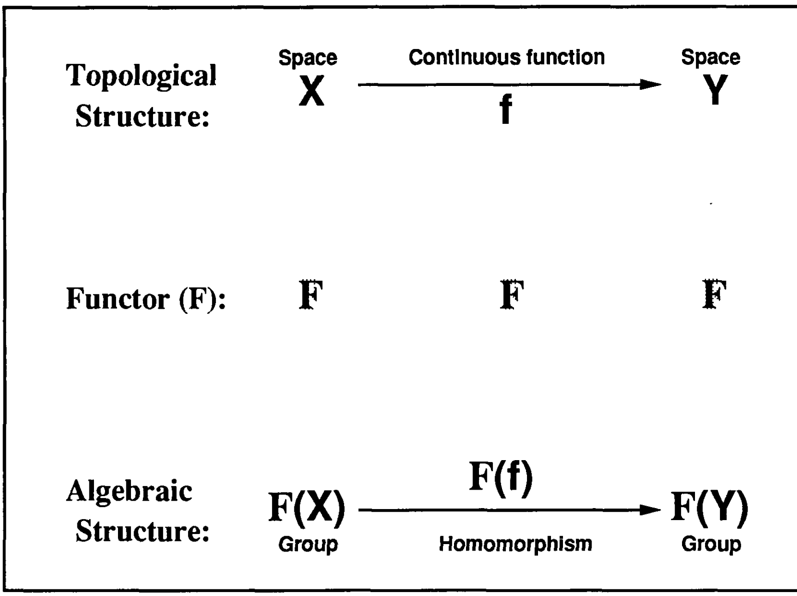
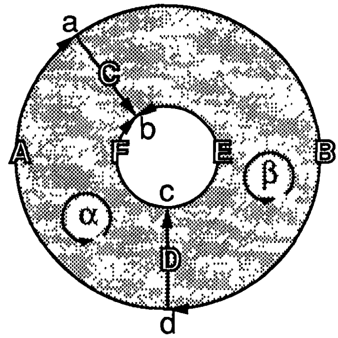


Figure 1: A functor converts topological structure to algebraic structure

Inevitably the algebraic invariants of topological spaces and topological functions cannot retain all of the topological information of the spaces and functions themselves. Often, for example, the algebraic objects are finite, or finitely generated and enumerable, while the interesting topological objects are uncountably infinite. Nonetheless, the reduction of information content to finite or finitely generated sets is precisely the transformation we need to operate with our mathematical model of a-map-as-a-continuum on a computer, which is a finite machine. The map, which has infinitely many points, is partitioned into finitely many cells, which we call 0-cells, 1-cells, and 2-cells depending on their dimension. Those finitely many cells are used to build algebraic structures called chain groups, one group for each relevant dimension; and algebraic boundary operators (homomorphisms) are defined between those groups which capture the essential topological boundary relations among the 0-cells, 1-cells, and 2-cells. Each element of the n-dimensional chain group is a formal linear combination of independent symbols, one symbol for each different n-cell.

0-cells: a, b, c, d
1-cells: **A, B, C, D, E, F**
2-cells: α, β



Boundary Operators:

$$\partial\alpha = \mathbf{D} + \mathbf{F} - \mathbf{C} - \mathbf{A}$$
$$\partial\beta = \mathbf{C} - \mathbf{E} - \mathbf{D} - \mathbf{B}$$

$$\partial\mathbf{A} = a - d \qquad \partial\mathbf{B} = d - a \qquad \partial\mathbf{C} = b - a$$
$$\partial\mathbf{D} = c - d \qquad \partial\mathbf{E} = b - c \qquad \partial\mathbf{F} = b - c$$
$$\partial a = \partial b = \partial c = \partial d = 0$$

Figure 2: Cell decomposition of annular region, associated group generators, boundary operators, and typical elements

The chain groups and boundary homomorphisms depend on the choice of cell decomposition of the space; and a map may usually be decomposed into cells in various ways.

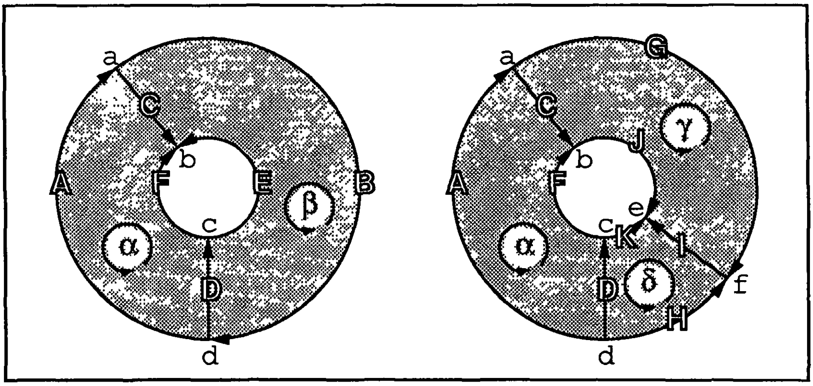
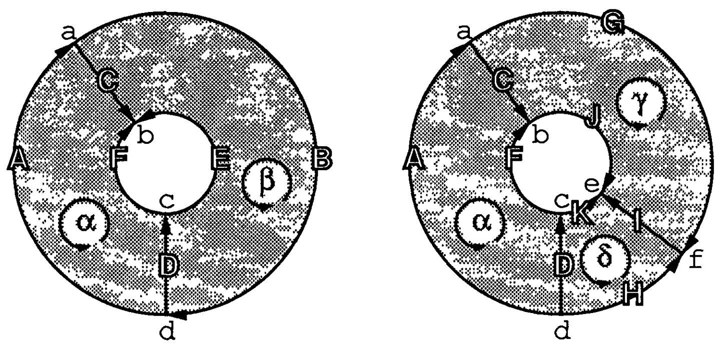


Figure 3: Two different cell decompositions of a region

2.3. Building Composite Algebraic Structures From Elementary Algebraic Structures on Topological Spaces

New groups, called homology groups, may, in turn, be derived from the chain groups by forming quotient groups of distinguished subgroups of cycles and boundaries of the chain groups. These homology groups surprisingly do not depend on the cell decomposition of the topological space, but on the space itself! That is, two different cell decompositions of the same space will produce two different chain groups, but the distinguished subgroups of the two chain groups will always, in turn, produce the same (up to isomorphism) collection of homology groups.



0-cells:	a,b,c,d	a,b,c,d,e,f
1-cells:	**A,B,C,D,E,F**	**A,C,D,F,G,H,I,J,K**
2-cells:	α,β	α,δ,γ

Generator of 1-dimensional homology group:

[A+B]=[F-E]	**[A+G-H]=[F+J-K]**
because	**because**
A+B=F-E$-\partial\{\alpha+\beta\}$	**A+G-H=F+J-K**$-\partial\{\alpha+\delta+\gamma\}$

Figure 4: Different cell decompositions yield same homology

Now let's look at the underlying significance of homology groups, and we will describe without proof the structure of homology for many topological spaces, including plane graphs (i.e. the linework of our cartographic objects).

2.4. Some Examples of Homology Groups

Homology groups describe the connectivity structure of the topological space. For maps represented by a full complement of 0-cells, 1-cells and 2-cells, the homology groups are uninterestingly trivial because the full cell structure adds up to a space which is topologically trivial–i.e. equivalent to a rectangle or (if it is a world map) equivalent to a sphere. All homology groups of a rectangle are 0 except the 0-dimensional group, which is Z, a single copy of the integers. We write $H_0(R) = Z$.

For the sphere, we have $H_0(S) = H_2(S) = Z$, and for all i different from 0 and 2, $H_i(S) = 0$.

The somewhat more useful homology groups are those of the line network (sometimes called the 1-skeleton) of the map. The 1-dimensional homology group measures simple connectivity (or lack thereof) of the topological space; and the graph network has many cycles and thus is not simply connected ("Simply connected" means that any loop can be shrunk continuously to a point without leaving the space.) The plane and the sphere are both simply connected. The annular region of figures 2 to 4 is not simply connected, hence H_1 of that region is not 0.

The following are useful summaries of how homology groups behave for the line graph network of a map and what they show about that network:

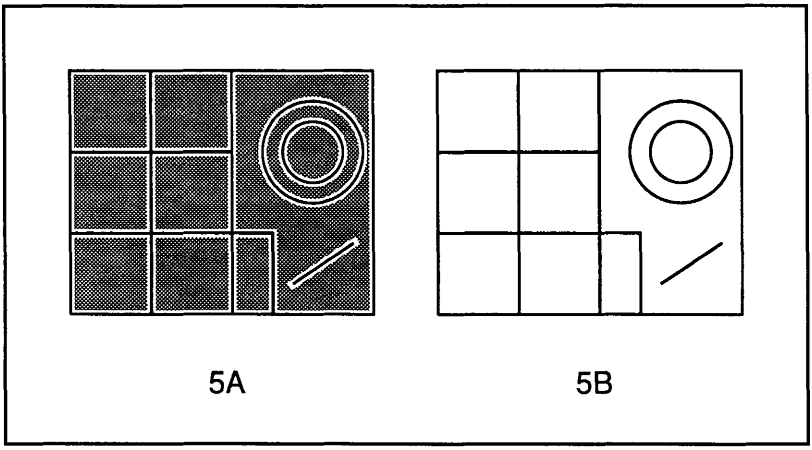


Figure 5: A map (A) and its line graph network (B).

For a topological space consisting of the linework of a planar graph (such as shown in figure 5B), the homology groups have the following structure:

$H_0(X) = Z \oplus Z \oplus Z \oplus \ldots \oplus Z \oplus Z$, n copies of Z, the integers, where n is the number of connected components of X. In the case shown in figure 5B, $n = 4$.

$H_1(X) = Z \oplus Z \oplus Z \oplus \ldots \oplus Z \oplus Z$, m copies of Z, the integers, where m is the maximum number of independent cycles of the graph X ("cycles" in the graph-theoretic sense, "independent" in the algebraic sense–no non-trivial linear combinations of these elements are zero.) In the case shown in figure 5B, $m = 10$, and a collection of generators for those cycles (in the graph sense) would be sums of the appropriately signed edges making up the outer boundaries of the ten regions shown in figure 5A. Notice that there are far more than 10 different cycles on the graph. What the homology group captures with its algebraic structure is the dependence relations of all of those infinitely many cycles. The homology group is more than just a count of how many independent cycles there are!

$H_i(X) = 0$ for all $i > 1$ since the line graph X has no 2-dimensional or higher dimensional elements which might generate cycles in the homology sense.

Loosely speaking, then, H_0 counts connected components of the line network, and H_1, though it is the homology group of the line network itself, also counts fundamental (interior) regions delimited by the line network.

3.1. Semi-exact Sequences and Exact Sequences

Algebraists have developed a standardized shorthand notation to describe essential structure of interesting subgroups and quotient groups: They have converted objects into homomorphisms and into sequences of homomorphisms in order to treat objects and homomorphisms with the same tools and operators. The tools focus on two important subgroups of a homomorphism, the kernel (ker) and the image (Im).

If $\Phi : G \longrightarrow K$ is a homomorphisms of groups then:

$$\ker(\Phi) = \{g \in G | \Phi(g) = e_K, \text{the identity of } K\}$$

and

$$\mathrm{Im}(\Phi) = \{k \in K | k = \Phi(g) \text{for some } g \in G\}.$$

If a sequence of two or more homomorphisms may be composed with each other because the appropriate domains and ranges match, then we may examine the relation of the image of a homomorphisms to the kernel of its successor:

$$\xrightarrow{\Phi_{i+1}} G_i \xrightarrow{\Phi_i} G_{i-1} \xrightarrow{\Phi_{i-1}} G_{i-2} \xrightarrow{\Phi_{i-2}} G_{i-3} \xrightarrow{\Phi_{i-3}}$$

If the image $\mathrm{Im}(\Phi_{i-k})$ is contained in the kernel $\ker(\Phi_{i-k-1})$ for all meaningful values of k, then we say that the above sequence is semi-exact.

If the image $\mathrm{Im}(\Phi_{i-k})$ is equal to the kernel $\ker(\Phi_{i-k-1})$ for all meaningful values of k, then the sequence is exact.

The two fundamental results on sequences of chain groups and induced groups, given without proof, are the following:

1. The boundary operators for chain groups always yield semi-exact sequences.

Elements that lie in the kernel of a boundary operator have zero boundary; and we call them cycles. Elements that lie in the image of the boundary operator are called boundaries (because they are boundaries of something!) Cycles that are not boundaries generate the homology groups, which describe the extent to which the semi-exact sequences induced by the boundary operators fail to be exact.

2. Homology groups may be embedded in natural exact sequences whose homomorphisms are induced by the boundary operators and inclusion maps

One such exact homology sequence is the Mayer-Vietoris Exact Homology Sequence described in the next section.

3.2. The Mayer-Vietoris Exact Homology Sequence

The Mayer-Vietoris Exact Homology Sequence relates the homology groups of the union and intersection of two "nice" topological spaces to the homology groups of the spaces themselves by embedding all the groups in an exact sequence:

$$\cdots \longrightarrow H_i(A \cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(A \cup B) \xrightarrow{\partial} H_{i-1}(A \cap B) \longrightarrow \cdots$$

Knowing that a sequence is exact, and knowing some of its groups, one may often deduce the missing groups. That is the approach that this exposition will utilize. We will not worry about the way in which the exact sequence is defined. The interested reader is referred to Hu [2] for a full explanation of the Mayer-Vietoris Sequence and sufficient conditions on the topological spaces A and B to guarantee exactness of the sequence.

4. USEFUL PROPERTIES OF HOMORPHISMS AND EXACTNESS

4.1. Rank of a commutative group

All of our homology groups are commutative and are finitely generated. Suppose that we have any commutative group that is finitely generated. Then the theory of groups tells us that the commutative group may be regarded as a direct sum of a number n of copies of the integers Z, $Z \oplus Z \oplus Z \oplus \ldots \oplus Z \oplus Z$, plus T, the torsion or finite subgroup of the larger group consisting of all elements of finite period.

The value n totally and uniquely determines the algebraic structure of the torsion-free part of this direct sum. The number n is called the rank of the group; and for any group homomorphism Φ, the rank has the following nice additive property:

$$\Phi : G \longrightarrow K$$

$$\operatorname{rank}(G) = \operatorname{rank}(\ker(\Phi)) + \operatorname{rank}(\operatorname{Im}(\Phi))$$

We will use this property to prove an important lemma.

4.2. Telescoping Lemma

The next lemma is the key to constructing a missing group in an exact sequence of groups:

Lemma: Suppose that the sequence given below is exact and that each group G_i has rank n_i.

$$0 \xrightarrow{\Phi_{i+n}} G_n \xrightarrow{\Phi_n} G_{n-1} \xrightarrow{\Phi_{n-1}} \cdots \xrightarrow{\Phi_3} G_2 \xrightarrow{\Phi_2} G_1 \xrightarrow{\Phi_1} 0$$

where Φ_{n+1} and Φ_1 are the zero homomorphisms. Then consider the following alternating sum:

$$\begin{aligned}
(-1)^n\operatorname{rank}(G_n) + (-1)^{n-1}\operatorname{rank}(G_{n-1}) + \cdots + \operatorname{rank}(G_2) - \operatorname{rank}(G_1) &= \sum_{i=1}^{n}(-1)^i\operatorname{rank}(G_i) \\
&= \sum_{i=1}^{n}(-1)^i n_i
\end{aligned}$$

Then this sum is zero by the exactness of the sequence.

Proof of the lemma:

Call $\operatorname{rank}(\ker(\Phi_i))$"$k_i$" and call $\operatorname{rank}(\operatorname{Im}(\Phi_i))$"$I_i$".

Let $k_0 = \operatorname{rank}(\ker(\Phi_0)) = I_1 = 0$ to simplify notation.

$$\begin{aligned}
\text{For } i > 0, \text{ each } \operatorname{rank}(G_i) &= \operatorname{rank}(\ker(\Phi_i)) + \operatorname{rank}(\operatorname{Im}(\Phi_i)) \\
&= k_i + I_i \\
&= \operatorname{rank}(\ker(\Phi_i)) + \operatorname{rank}(\ker(\Phi_{i-1})) \\
&= k_i + k_{i-1}
\end{aligned}$$

$$\text{Thus:} \qquad \sum_{i=1}^{n}(-1)^i \text{rank}(G_i) = \sum_{i=1}^{n}(-1)^i n_i$$
$$= \sum_{i=1}^{n}(-1)^i (k_i + k_{i-1})$$

But the alternating sum causes all terms to cancel except possibly:

$$k_n + I_1 = \text{rank}(\ker(\Phi_n)) - \text{rank}(\text{Im}(\Phi_1))$$

But by exactness, $\ker(\Phi_n) = \text{Im}(\Phi_{n+1}) = 0$, and $\text{Im}(\Phi_1) = 0$. For consistency, we let $\text{rank}(0) = 0$.

Next we see why rank is useful to know.

5. APPLYING THE RESULTS TO THE OVERLAY PROBLEM

Now let's put some of our results together. We know some homology groups. We have seen one exact sequence, the Mayer-Vietoris Sequence, which relates homology groups for two spaces, their union, and their intersection. Finally we have the *telescoping lemma* which allows us to relate in a single equation the ranks of all of the homology groups that appear in an exact sequence. We merely need to observe how we can actually calculate the ranks of all but one of the homology groups that appear in the Mayer-Vietoris Sequence, and then we will know the remaining group's rank.

Let A and B be two line graphs of maps to be overlaid. Then the portion of the Mayer-Vietoris sequence that may contain non-zero entries is the following:

$$\cdots \to H_2(A \cup B) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(A \cup B) \to$$
$$\to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(A \cup B) \to H_{-1}(A \cap B) \to \cdots$$

where both $H_2(A \cup B)$ and $H_{-1}(A \cap B)$ are zero.

The term in the sequence that we want to compute is $H_1(A \cup B)$; and we can find that term by examining $A \cap B$, the intersection graph. Standard graph traversal methods allow us to detect all common components of A and B and to find their intersections. All that remains is to describe $A \cap B$ in terms of its number of disconnected components and its number of independent cycles. Again standard graph traversal techniques permit us to derive these numbers.

Then we know from the Telescoping Lemma that:

$$\text{rank}(H_1(A \cap B)) - \text{rank}(H_1(A) \oplus H_1(B)) + \text{rank}(H_1(A \cup B)) -$$
$$\text{rank}(H_0(A \cap B)) + \text{rank}(H_0(A) \oplus H_0(B)) - \text{rank}(H_0(A \cup B)) = 0.$$

Furthermore, the rank of a direct sum is just the sum of the ranks:

$$\text{rank}(H_i(A) \oplus H_i(B)) = \text{rank}(H_i(A)) + \text{rank}(H_i(B))$$

Finally, recall that the $\text{rank}(H_1(X))$ is just a count of the interior regions separated by the line graph X; and $\text{rank}(H_0(X))$ is simply the number of components of X. Putting it all together, and using the notation:

$$r(X) = \text{rank}(H_1(X)) \text{and } c(X) = \text{rank}(H_0(X)),$$

we get:

$$r(A \cap B) - (r(A) + r(B)) + r(A \cup B) - c(A \cap B) + (c(A) + c(B)) - c(A \cup B) = 0$$

Notice further that if $r'(X)$ represents the total number of regions of the map (not just the interior regions), then the equation still holds (because $r'(X) = r(X) + 1$, and r appears twice with a plus sign and twice with a negative sign):

$$r'(A \cap B) - (r'(A) + r'(B)) + r'(A \cup B) - c(A \cap B) + (c(A) + c(B)) - c(A \cup B) = 0$$

Isolating $r(A \cup B)$ (or r') we get:

$$r(A \cup B) = r(A) - c(A) + r(B) - c(B) - (r(A \cap B) - c(A \cap B)) + c(A \cup B)$$

We conclude with the example in figure 6 to illustrate our methods.

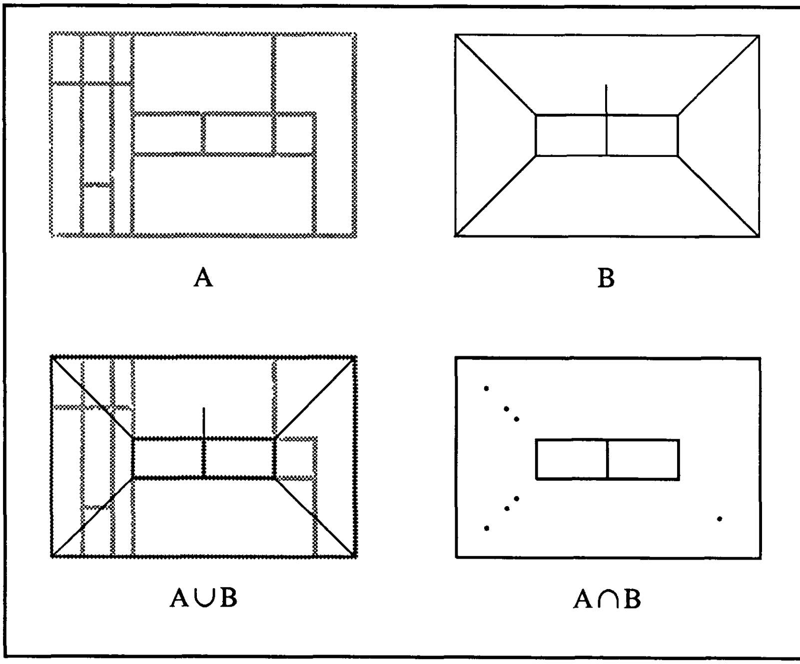


Figure 6: Deriving the Complexity of Overlaying A and B

In figure 6 we see that $r(A) = 13$, $r(B) = 6$, and $r(A \cap B) = 3$. Moreover, because A, B, and $A \cup B$ are all connected, $c(A) = c(B) = c(A \cup B) = 1$. Finally, the number of components of $A \cap B$, $c(A \cap B)$, is 9. Thus by our formula $r(A \cup B) = 24$.

We see from our example that a critical contributor to the sum on the right is the term $c(A \cap B)$, the number of new components (usually isolated intersections) of the intersection graph. By our formula, every new intersection gives rise to a new region! This observation may be useful in estimating the number of new regions that arise in overlay operations. If, for example, we can place a bound on the number of new intersections that will occur, then we can conclude that the number of new regions will be bounded accordingly This is a nice duality relation that we will develop in a later, longer paper.

6. CONCLUSIONS

We have introduced a few useful ideas from the realm of algebraic topology in order to illustrate one way of applying important duality relations to a specific combinatorial problem. In effect we have converted the problem of determining the number of regions arising from polygon overlay to a graph traversal and intersection detection problem. Further research is planned along the following lines:

1. Describe properties of the line segments in the line networks to be overlaid (such as extent, density, etc.) that would produce a guaranteed bound on the number and type of intersections and a corresponding bound on the number of new regions created.

2. Integrate topological information into the computation of the intersection graph in order to prevent slivers, gaps, and other anomalies due to geometric imprecision.

3. Develop relative homology groups for analysis of local combinatorial duality relationships.

I will write up new results and elaboration of the results sketched here in a more extensive research paper.

7. REFERENCES

1. HENLE, MICHAEL, 1979, A Combinatorial Introduction to Topology, W. H. Freeman and Company, San Francisco.

2. HU, SZE-TSEN, 1966, Homology Theory: A First Course in Algebraic Topology, Holden-Day, Inc., San Francisco.